Technical report #6
on
Controlling the synchronous machines

presented to

Professor Katsuhisa Furuta
Furuta Laboratory (of Control System Technology)
Department of Mechanical and Environmental Informatics
Tokyo Institute of Technology

Kittisak Tiyapan[1]
kittisak@ctrl.titech.ac.jp

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Its frequency in cyscles per second (hertz) is the same as the speed of the rotor in revolutions per second; i.e., the electrical frequency is synchronized with the mechanical speed, and this is the reason for the designation *synchronous* machine. Thus a 2-pole synchronous machine must revolve at 3,600 rpm to produce a 60-Hz voltage.

Electric Machinery 3^{rd} Edition, Fitzgerald et. al. [FCKK71].

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Introduction

In this Technical Report #6 synchronous motors were studied and control schemes were developped for them. Each Technical Report of this series will be written approximately 100 pages long.

To Professor Furuta,

Prof., you may use any material in this report or any other similar reports which are written for you, for any purpose you'd like, with or without acknowledging me.

Kittisak Tiyapan

Preliminary knowledge

The electrical frequency of a synchronous machine synchronizes with the mechanical speed. However it is more convenient to express angles in electrical unit (electrical angle) rather than in mechanical unit (mechanical angle) because of the reason that a synchronous machine can have more than 2 poles. Whenever this is the case, one can consider only a single pair of poles and consider those conditions, namely the electric, magnetic, and mechanical conditions, which are associated with all of the other pole pairs as repetitions of those for the pair under consideration. [FCKK71]
Electrical and mechanical angles are related by the relation

$$\theta_e = \frac{p}{2}\theta_m, \tag{1}$$

where, for a generator, θ_e is the electrical angle of the output, θ_m is the angle of the rotating shaft, and p is number of poles of the machine. The frequency of the voltage wave is

$$f = \frac{p}{2}\frac{n}{60} \quad \text{Hz}, \tag{2}$$

where n is the mechanical speed in rpm and $\frac{n}{60}$ is the speed in revolutions per second. The radian frequency ω of the voltage wave is

$$\omega_e = \frac{p}{2}\omega_m, \tag{3}$$

where ω_m is the mechanical speed in radians per second.
Figure 1 shows a juxtaposition between mechanical and electrical rotation of a synchronous machine which has got three pairs of poles.
In power system engineering the two most important things to be controled are power and frequency. Important components of the control are for example the speed governor, regulators, actuators and signal transducers. The speed governor can be described by

$$\Delta f \cdot K = T_L, \tag{4}$$

where Δf is the change in frequency, K a gain and T_L the load torque. The rest of the components mentioned are high-gain power amplifiers which convert position, oil pressure, electricity, etc. into valve positions.
A schematic diagram of power-frequency control is shown in Figure 2 ([Sin87], volume 2 pages 1375-1380). In Figure 2 P_s is the set-point power, P the measured power, f_s the set-point frequency, f the measured frequency, $\mathbf{R}$ is the regulator, $\mathbf{T}$ the turbine, $\mathbf{G}$ the generator and $\mathbf{K}$ the feedback gain.

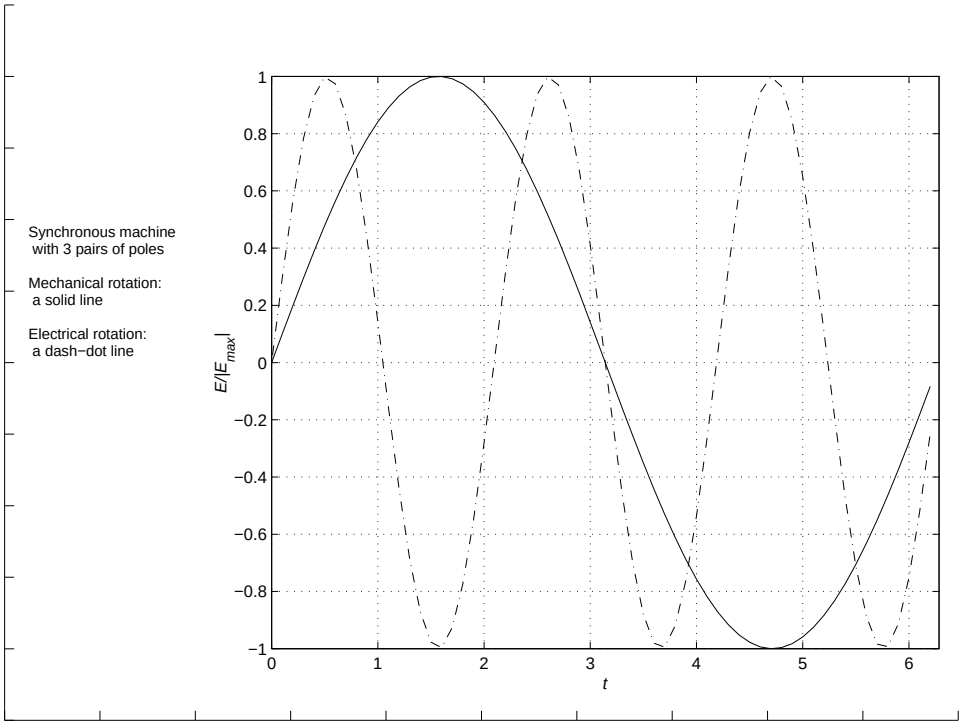


Figure 1: *Dynamic of mechanical and electrical angles*

At stationary operating points the error between measured and set-point values is zero, ie

$$\begin{aligned} e = 0 &= (P_s - P) + K(f_s - f), \\ f &= f_s + \frac{P_s - P}{K}, \\ \frac{df}{dP} &= -\frac{1}{K} = -D, \end{aligned} \tag{5}$$

where D is called the droop and has a unit of Hz/MW. This droop D is usually described as a dimensionless parameter D^*, where

$$D\frac{P_0}{f_0} = \frac{\left(\frac{\Delta f}{f_0}\right)}{\left(\frac{\Delta P}{P_0}\right)} = D^*. \tag{6}$$

Figure 3 show a typical graph between frequency and power, showing the drooping characteristic. When the load changes the straight line moves up or down but the slope stays the same for each synchronous generator.
The change in load can be either one of the following,

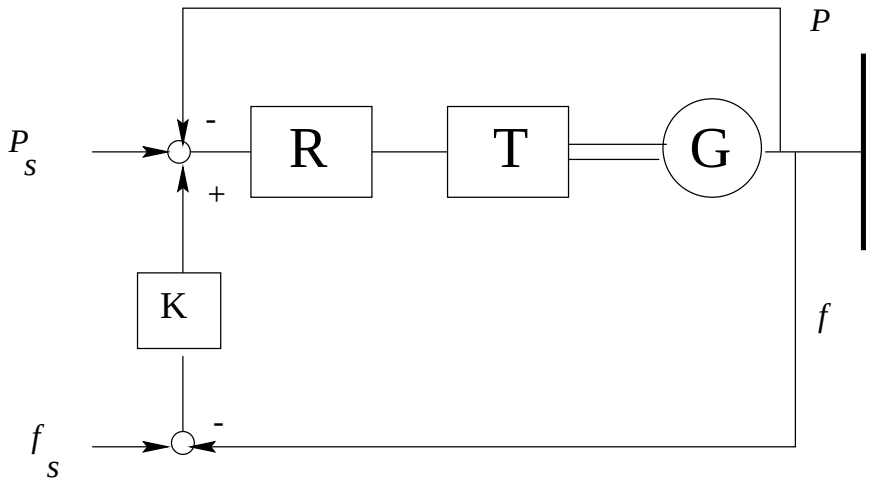


Figure 2: *Power-frequency control schematic diagram of a synchronous generator*

1. *load shedding,* the load decreases which results in the increase in frequency,
2. *generator shedding,* the load increases which results in the decrease in frequency,
3. *short-circuit,* generator is suddenly disconnected and the frequency will increase to the maximum frequency.

These effects caused by the load change can be related to

$$J\frac{d^2\theta}{dt^2} = T_{PM} - T_E, \tag{7}$$

where J is the moment of inertia of the rotating mass, θ the angular position of the rotating mass, T_{PM} is the prime-mover torque and T_E the electrical torque.

The Basic Electromechanical Equation

The electromechanical equation for a synchronous machine comes from the equation of torques acting on the rotating members [FCKK71]

$$T_{inert.} + T_{emag.} = T_{mech.}, \tag{8}$$

where $T_{inert.}$ is the inertia torque, $T_{emag.}$ an electromagnetic torque, and $T_{mech.}$ represents the input from the prime mover (generator) or the output to turn the load (motor).
Equation 8 is generator-conventioned.

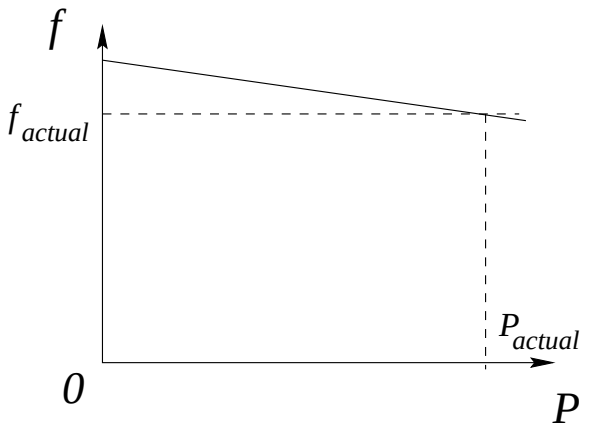


Figure 3: *Frequency-power characteristic curve of a synchronous generator*

A similar equation to Equation 8 which is more practical to obtain is the equation of power

$$P_{inert.} + P_{emag.} = P_{mech.}. \tag{9}$$

Here

$$P_{inert.} = P_j \frac{d^2\delta}{dt^2}, \tag{10}$$

where [1]

$$\dot{\delta} = \dot{\delta}_e = \dot{\theta}_e - \omega_R = \dot{\theta}_e - \frac{2\pi n}{60} \tag{11}$$

$$P_j = J\frac{2}{p}\frac{2\pi n}{60} \approx J\frac{2}{p}\omega_R \tag{12}$$

J is the moment of inertia, n the speed in rpm, the synchronous speed

$$\omega_R = \omega_m|_{rated} = \dot{\theta}_m\Big|_{rated}. \tag{13}$$

For simple problems

$$\delta \approx \text{the power angle of the machine.} \tag{14}$$

The electromagnetic power is

$$P_{emag..} = \text{damping power } + \text{ synchronous power} \tag{15}$$

$$= P_d\frac{d\delta}{dt} + P(\delta), \tag{16}$$

[1] $\theta = \omega_R t + \delta + \frac{\pi}{2}$ ([AF77], page 116)

where

$$P_d \approx \text{constant}. \tag{17}$$

The term $P(\delta)$ for a single machine on an infinite bus can be considered as follows.

1. When saliency is neglected, the equivalent circuit and phasor diagram is shown in Figure 4. From Figure 4

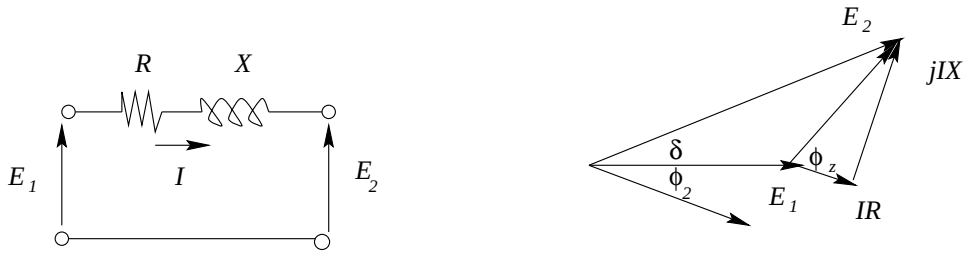


Figure 4: *Equivalent circuit and phasor diagram when saliency is neglected*

$$I \quad = \frac{E_1 \angle(\delta) - E_2 \angle(0)}{Z \angle(\phi_z)} \quad = \frac{E_1}{Z} \angle\left(\delta - \phi_z\right) - \frac{E_2}{Z} \angle\left(-\phi_z\right), \tag{18}$$

where

$$Z \angle\left(\phi_z\right) = R + jX. \tag{19}$$

Then

$$I \cos\phi_z = \frac{E_1}{Z} \cos\left(\delta - \phi_z\right) - \frac{E_2}{Z} \cos\left(-\phi_z\right), \tag{20}$$

and

$$\begin{aligned} P_2 &= E_2 I \cos\phi_2 \\ &= \frac{E_1 E_2}{Z} \cos\left(\delta - \phi_z\right) - \frac{E_2^2}{Z} \cos\left(-\phi_z\right) \\ &= \frac{E_1 E_2}{Z} \cos\left(\delta - \phi_z\right) - \frac{E_2^2 R}{Z^2} \qquad (21) \\ &= \frac{E_1 E_2}{Z} \sin\left(\delta + \alpha_z\right) - \frac{E_2^2 R}{Z^2}, \qquad (22) \\ & \qquad (23) \end{aligned}$$

where

$$\alpha_z \quad = \frac{\pi}{2} - \phi_z \quad = \tan^{-1}\frac{R}{X}. \tag{24}$$

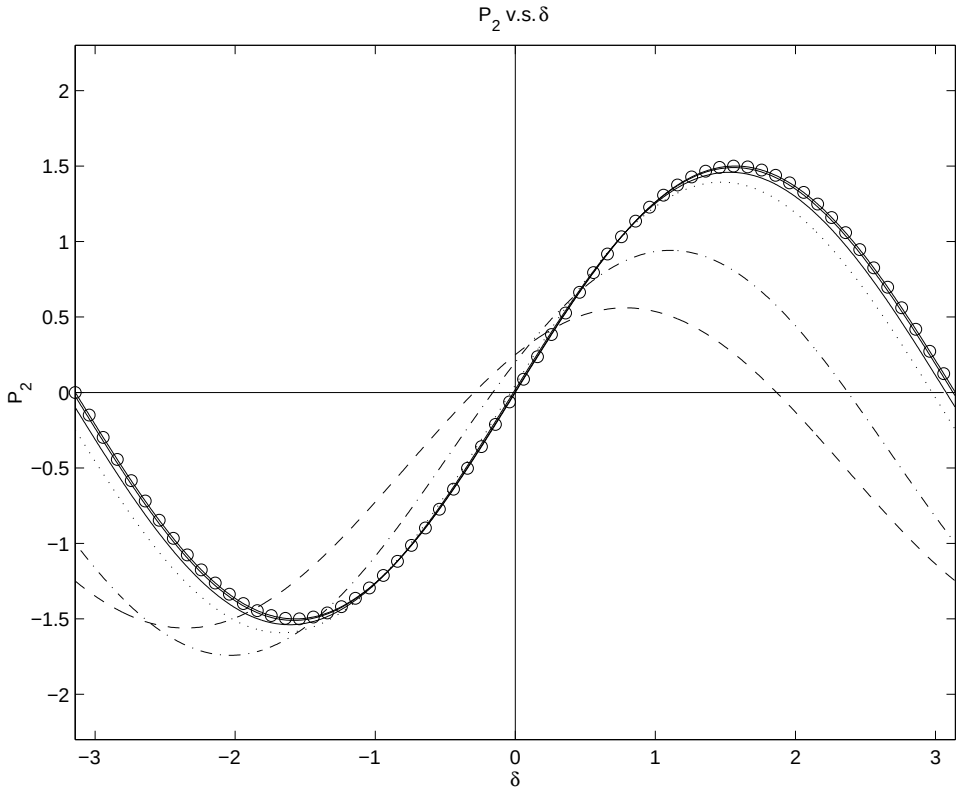


Figure 5: *Power-angle characteristic at load end ($R = 0$,0.01, and 0.04 are solid lines, $R = 0.1$ dotted line, $R = 0.5$ dash-dotted line, $R = 1$ dashed line, circles are drawn to emphasize $R = 0$)*

Figure 5 shows an example of a plot of P_2 as related to δ. Here we let

$$E_1 = 1.5, \quad E_2 = 1, \quad X = 1, \quad R = 0, 0.01, 0.04, 0.1, 0.5, 1.$$

Figure 6 shows an example of a plot of P_2 as related to δ with various values of E_2. Here we let

$$E_1 = 1, \quad X = 1, \quad R = 0.04, \quad E_2 = 0, 0.5, 1, 1.5, 2.$$

At source end

$$P_1 = \frac{E_1 E_2}{Z} \sin(\delta - \alpha_z) + \frac{E_2^2 R}{Z^2}. \tag{25}$$

Figure 7 shows an example of a plot of P_1 as related to δ. Here we let

$$E_1 = 1.5, \quad E_2 = 1, \quad X = 1, \quad R = 0, 0.01, 0.04, 0.1, 0.5, 1.$$

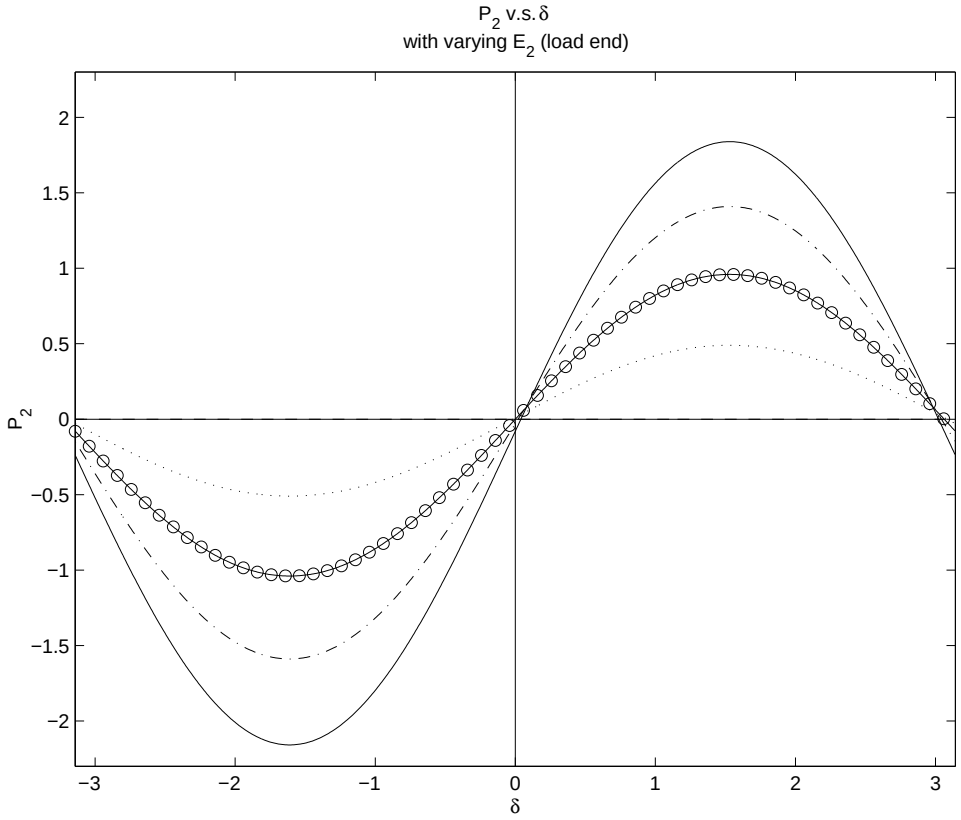


Figure 6: *Power-angle characteristic at load end ($E_2 = 0$ dashed line, $E_2 = 0.5$ dotted line, $E_2 = 1$ solid line and circles, $E_2 = 1.5$ dashed-dotted line, and $E_2 = 2$ also a solid line.)*

Figure 8 shows an example of a plot of P_1 as related to δ with various values of E_2. Here we let

$$E_1 = 1, \quad X = 1, \quad R = 0.04, \quad E_2 = 0, 0.5, 1, 1.5, 2.$$

Provided that

$$R \ll X, \tag{26}$$

we have

$$P_1 \quad = P_2 \quad = \frac{E_1 E_2}{X} \sin\delta. \tag{27}$$

If

$$E_1 = \text{constant}, \quad \text{and} \quad E_2 = \text{constant}, \tag{28}$$

then

$$P_{1,max.} \quad = P_{2,max.} \quad = \frac{E_1 E_2}{X} \sin\frac{\pi}{2} \quad = \frac{E_1 E_2}{X}. \tag{29}$$

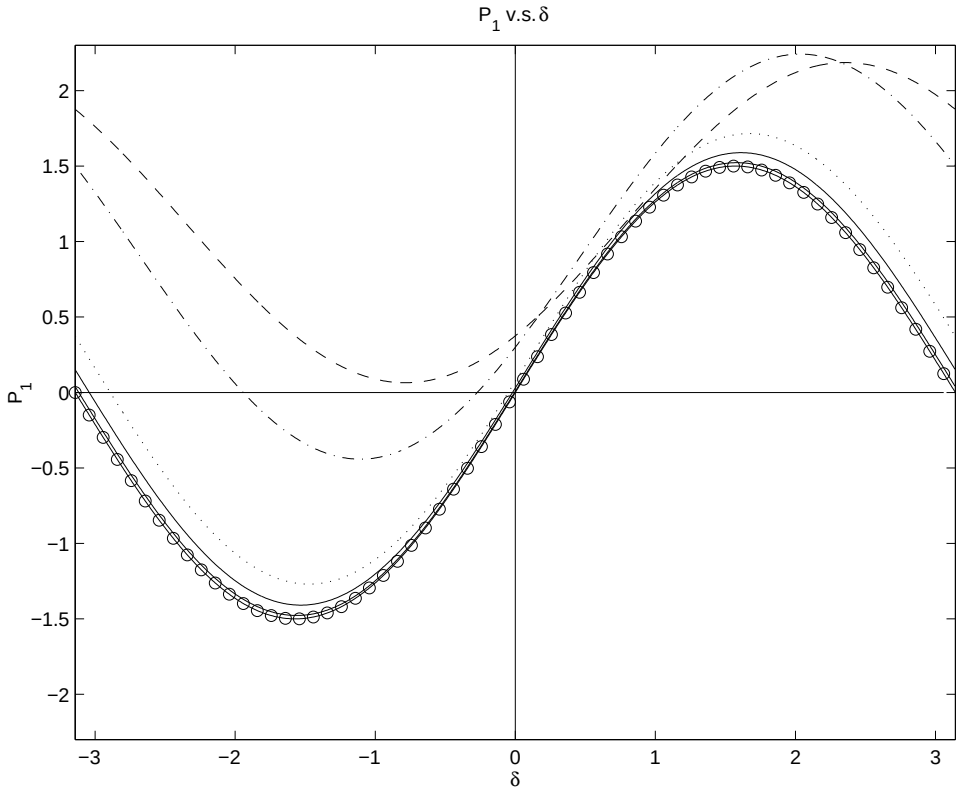


Figure 7: *Power-angle characteristic at source end ($R = 0$,0.01, and 0.04 are solid lines, $R = 0.1$ dotted line, $R = 0.5$ dash-dotted line, $R = 1$ dashed line, circles are drawn to emphasize $R = 0$)*

2. For salient pole machines, the equivalent circuit and phasor diagram is shown in Figure 9. Then

$$X_d = x_d + x_e \tag{30}$$
$$X_q = x_q + x_e \tag{31}$$

The power delivered to the bus per phase is

$$P = I_d E_e \sin\delta + I_q E_e \cos\delta. \tag{32}$$

From Figure 9

$$I_d = \frac{E_f - E_e \cos\delta}{X_d} \tag{33}$$
$$I_q = \frac{E_e \sin\delta}{X_e} \tag{34}$$

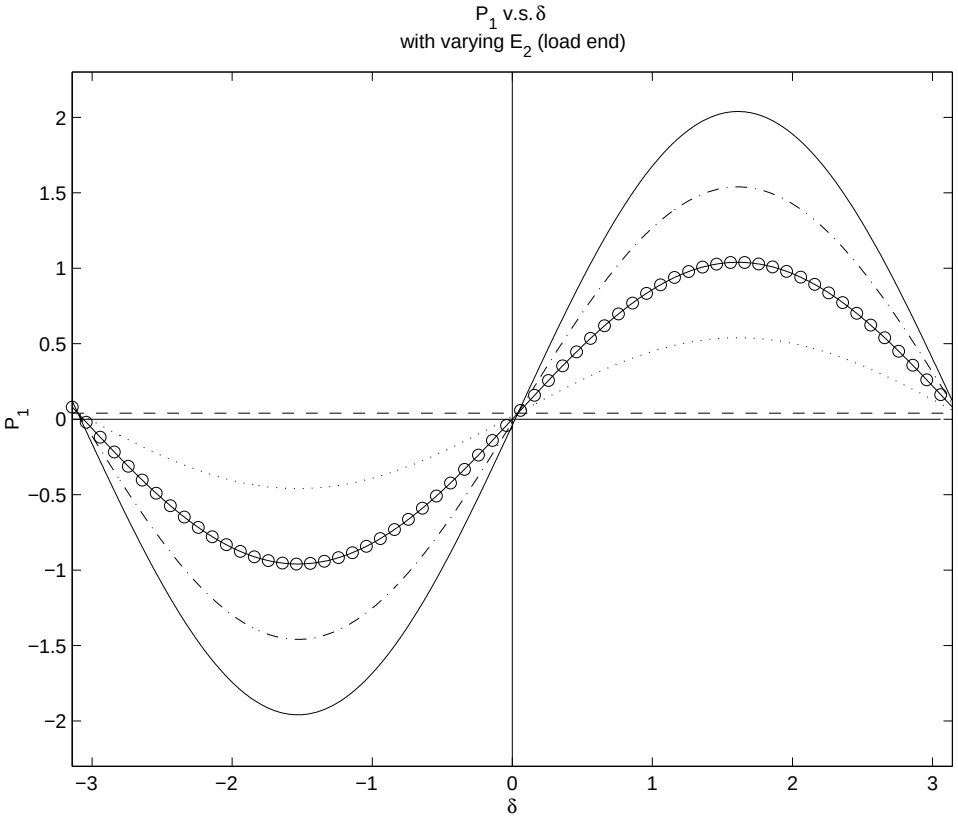


Figure 8: *Power-angle characteristic at source end ($E_2 = 0$ dashed line, $E_2 = 0.5$ dotted line, $E_2 = 1$ solid line and circles, $E_2 = 1.5$ dashed-dotted line, and $E_2 = 2$ also a solid line.)*

From Equation 32, 33, and 34,

$$P = \frac{E_f E_e}{X_d} \sin\delta + E_e^2 \frac{X_d - X_q}{2X_d X_q} \sin 2\delta \tag{35}$$

Figure 10 shows the power-angle characteristic for a salient-pole machine.

Table 1 shows a generator-convention and Table 2 a motor-convention. Throughout this report the generator-convention is adopted.

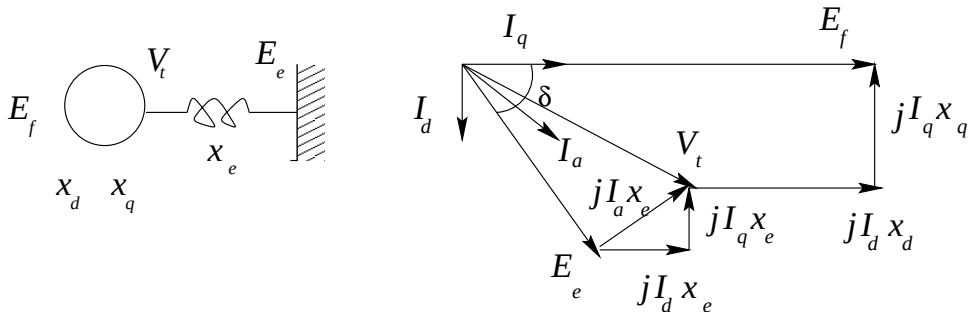


Figure 9: *Equivalent circuit and phasor diagram when saliency is taken into account*

Generator Convention		
$(\cdot)$	$(\cdot) < 0$	$(\cdot) > 0$
δ	motor action	generator action
$\frac{d\delta}{dt}$	$\dot{\theta} < \omega_R$	$\dot{\theta} > \omega_R$
$\frac{d^2\delta}{dt^2}$	deceleration	acceleration
$P_{mech.}$	output from the shaft	input to the shaft

Table 1: *Generator sign convention*

Motor Convention		
$(\cdot)$	$(\cdot) < 0$	$(\cdot) > 0$
δ	generator action	motor action
$\frac{d\delta}{dt}$	$\dot{\theta} > \omega_R$	$\dot{\theta} < \omega_R$
$\frac{d^2\delta}{dt^2}$	acceleration	deceleration
$P_{mech.}$	input to the shaft	output from the shaft

Table 2: *Motor sign convention*

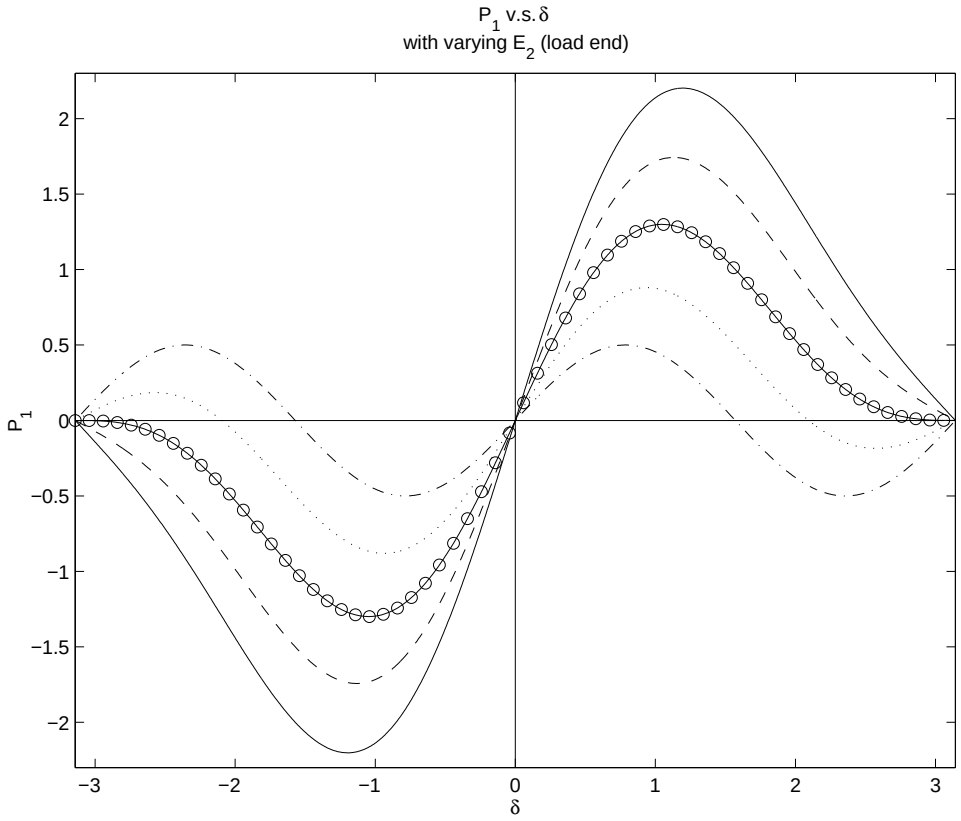


Figure 10: *Power-angle characteristic of a salient-pole machine ($R = 0$, $E_e = 1$, $X_d = 1$, $X_q = 0.5$, $E_f = 0$ dash-dotted line, $E_f = 0.5$ dotted line, $E_f = 1$ solid line and circles, $E_f = 1.5$ dashed line, $E_f = 2$ also a solid line)*

Discussion

Figure 5 and 7 show that the power-angle characteristic of a synchronous generator and that of a synchronous motor are alike only when

$$R \ll X.$$

(For the generator-convention, the right-hand side of the power-axis is for a generator while the left-hand side is for a motor.)
Because there is a nonlinearity of the sine function within the state equations, if we did state-transformation from $x_1 = \delta$, $x_2 = \dot{\delta}$ to, say $z_1 = \theta$, $x_2 = \dot{\theta}$, the state equations would have changed form, and thus a different kind of responses could be expected. Therefore it is important to keep in mind that the study of synchronous machines' dynamic concerns mainly with δ, not θ.
For a singly running synchronous generator, in the control point of view one is concerned with controlling δ such that

$$|\delta(t)| < \vartheta, \quad \forall t \in \mathcal{R}^+, \tag{36}$$

here a specified parameter ϑ is the tolerance of deviation in speed of the generator. There are hree situations to consider to ensure a good design, they are

1. during a normal run of the machine (here the objective is to maximize the utility and optimize the costs),

2. when the machine is subjected to load fluctuations (here the objective is for robustness), and

3. when a fault occurs (here the objective is for a fail-proof system).

In the case where more than one generators supplying electrical power to a network, another situation that must be considered is the situation when some of the machines are suddenly disconnected from the network. Here the remaining generators would see the disconnection as a kind of fault occuring withing the network.

Simplified model of a synchronous machine

The swing equation in the form most useful in practice is [AF77]

$$\frac{2H}{\omega_R}\dot{\omega} = T_a, \tag{37}$$

where

$$H \equiv \frac{W_k}{S_{B3}} \quad \text{s}, \tag{38}$$

S_{B3} is the rated three-phase MVA of the system,

$$W_k = \frac{1}{2}J\omega_m^2, \tag{39}$$

T_a is the accelerating torque (N·m),

$$J\ddot{\theta} = T_a \tag{40}$$

$$T_a = T_m - T_e. \tag{41}$$

The classical model of one synchronous machine connected to an infinite bus [AF77]

$$\frac{2H}{\omega_R}\frac{d^2\delta}{dt^2} = P_m - P_e \tag{42}$$

$$P_e = P_C + P_M \sin(\delta - \gamma). \tag{43}$$

When the shunt load at the machine terminal is open and where the transmission network is reactive

$$P_C = 0, \quad \gamma = 0. \tag{44}$$

A model of a synchronous electrical machine used in power system engineering is represented as

$$H\ddot{\delta} = P_m - P_e \sin\delta \tag{45}$$

where δ is the rotor angle relative to a reference frame, H is the inertia constant, P_m is the mechanical power supplied, and P_e is the maximum electrical power which can be generated. The machine is driven by a mechanical power source and supplies electrical power to an infinite busbar. In this model friction, magnetic saturation, variation of field strength and saliency are neglected. This equation can be put into state-space form as

$$\dot{x_1} = x_2 \tag{46}$$

$$\dot{x_2} = \frac{P_m - P_e \sin x_1}{H} \tag{47}$$

$$y = P_e \sin x_1 \tag{48}$$

where $x_1 = \delta$ and $x_2 = \dot{\delta}$. When $P_m > 0$ this machine acts as a generator and when $P_m < 0$ a motor. The equilibrium point ($f(x) = 0$) is at the point

$$(x_1, x_2) = \left(\sin^{-1}\frac{P_m H}{P_e}, 0\right)$$

The Jacobian matrix at the equilibrium points are

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\left(\sin^{-1}\left(\frac{P_m H}{P_e}\right),0\right)} = \left.\begin{bmatrix} 0 & 1 \\ -\frac{P_e}{H}\cos x_1 & 0 \end{bmatrix}\right|_{x_1=\sin^{-1}\frac{P_m H}{P_e}} = \begin{bmatrix} 0 & 1 \\ -\frac{P_e}{H}\sqrt{1-\frac{P_m^2 H^2}{P_e^2}} & 0 \end{bmatrix},$$

which has got eigenvalues of

$$\pm\frac{1}{H}\sqrt{\pm H\sqrt{P_e^2 - P_m^2 H^2}}$$

Desired response of the system

The response of an ideal generator should be as shown in Figure 11 always and for all kinds of loading (robust), a mathemetical description of which would be

$$\left.\begin{array}{ll} x_1 &= \theta \quad = \omega_0 t \\ x_2 &= \dot{\theta} \quad = \omega_0 \end{array}\right\}, \tag{49}$$

where x_1 and x_2 here represents absolute values of angle and angular velocity respectively, therefore are not the same as those mentioned in the previous section.
Figure 12 shows three kinds of undesirable response, in (a) the generator rotates to and fro resonantly, in (b) the generator stops and in (c) the
The undesirable response shown in Figure 12 could be compared with the responses of systems with known responses as shown in Figure 13. The systems from which Figure 13 was produced were

$$\left.\begin{array}{ll} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + a \end{array}\right\} \text{(a)}, \quad \left.\begin{array}{ll} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2\zeta x_2 + a \end{array}\right\} \text{(b)}, \quad \left.\begin{array}{ll} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 2\zeta x_2 + a \end{array}\right\} \text{(c)},$$

where $a = 2$ and $\zeta = 0.05$ here. $\theta(t)$ of a synchronous machine should not have a trajectory similar to any one of the three $x_1(t)$'s in Figure 12.
However, when studying the dynamic behaviour of a synchronous machine, the angle δ considered is actually the angle between the electrical degree of a point on the rotating shaft and a reference axis which is rotating at the synchronous speed, and δ is called the *torque angle*. Whenever the speed of a synchronous machine differs from the synchronous speed, the rotor speed is no longer synchronizing with the stator field, and

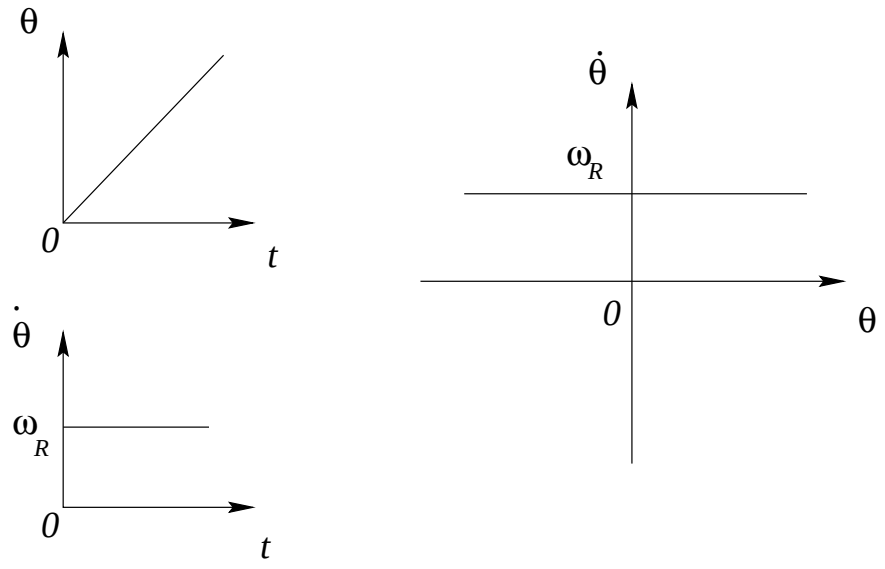


Figure 11: *Ideal response of the simplified synchronous generator*

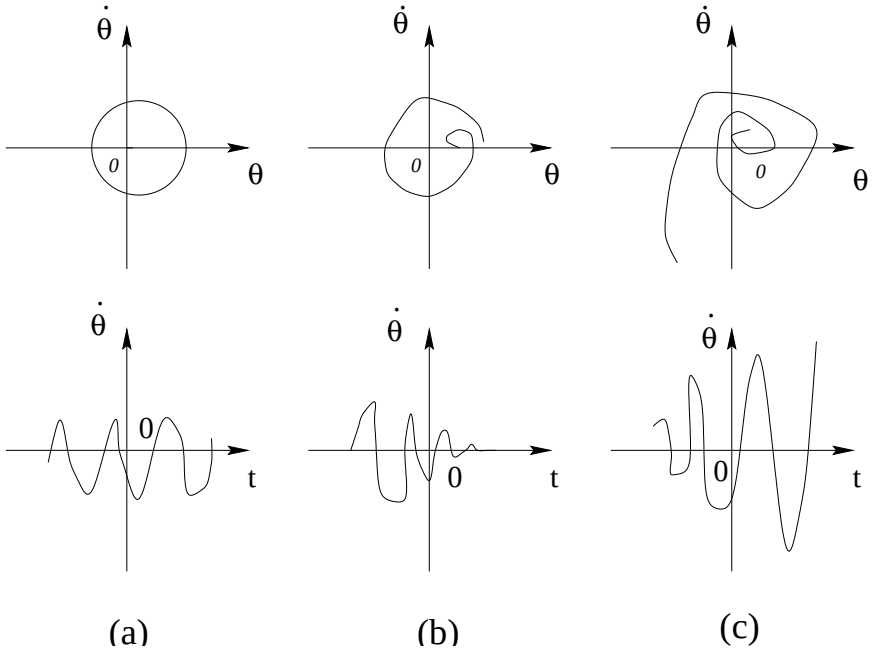


Figure 12: *Undesirable responses of the simplified synchronous generator*

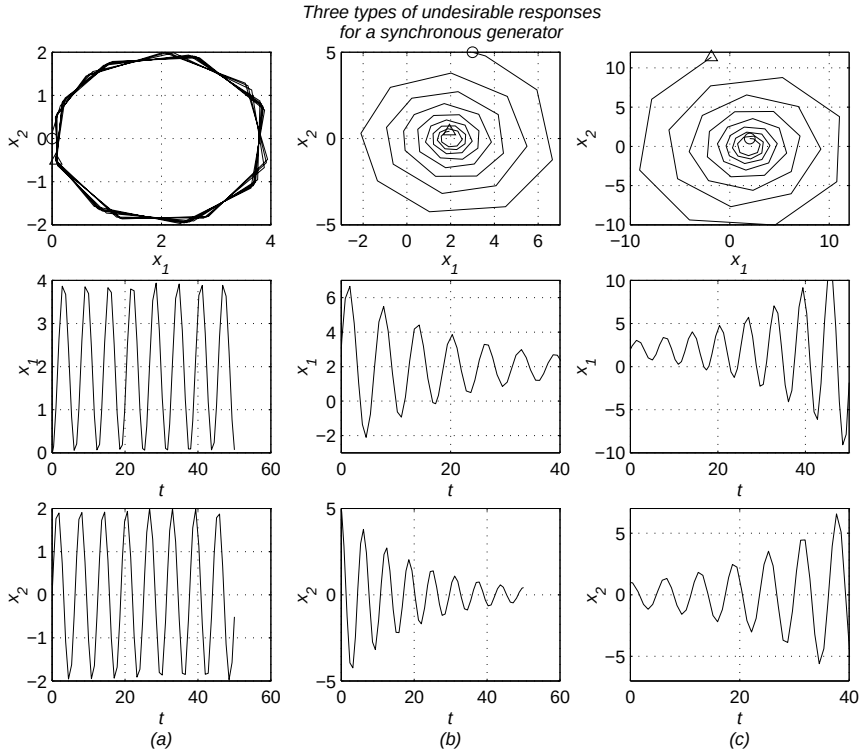


Figure 13: *Responses from known systems for comparison with the undesirable responses of the synchronous generator (a)* $\ddot{x} = -x + a$*, (b)* $\ddot{x} = -2\zeta x + a$*. and (c)* $\ddot{x} = 2\zeta x + a$*,* $a = 2, \zeta = 0.05$ *((b) is underdamped)*

Example of states trajectory

Figure 14 shows the state plane of a generator based on this model when

$$P_m = 2, \quad P_e = 1.2 \quad \text{and} \quad H = 1.3,$$

and the initial conditions were

$$(x_1, x_2) \in \{-5.5, -5.1, -4.7, 0.5, 0.9, 1.3, 1.7, 7, 7.4, 7.8\} \times \{0\} \cup \{-9, -8.6, -8.2. \ldots, 19.8\} \times \{-4.1\}$$

Total CPU time of simulation was $t_{CPU} = 0.7700$ seconds.
This means that the system is a generator, and

$$\left.\begin{array}{ll} \dot{x_1} & = x_2 \\ \dot{x_2} & = 1.5385 - 0.9231 \sin x_1 \\ y & = 1.2 \sin x_1 \end{array}\right\}. \tag{50}$$

Equilibrium points cannot be calculated for Equation 50 since setting $\dot{x} = f(x) = 0$ yields the solution of

$$\left(\sin^{-1}\left(\frac{2}{1.2}\right), 0\right) = (1.5708 - \jmath 1.0986),$$

therefore there is no equilibrium point to be found. The only thing we could say is that $x_2 = 0$. If we try to find the Jacobian matrix from

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\left(\sin^{-1}\left(\frac{2}{1.2}\right),0\right)} = \begin{bmatrix} 0 & 1 \\ -0.9231 \cos x_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.9231 \cos\left(\sin^{-1}\left(\frac{2}{1.2}\right)\right) & 0 \end{bmatrix},$$

the eigenvalues would then be

$$-0.7845 + \jmath 0.7845, \quad \text{and} \quad 0.7845 - \jmath 0.7845$$

which is quite an interesting pair of eigenvalues since the two of them are symmetric to the origin (as contrasted to the conventional cases where eigenvalues are normally symmetric to one another with respect to the x-axis.)
Figure 15 shows the state plane of a generator based on this model in a case where $P_m < P_e$. This time

$$P_m = 1.7, \quad P_e = 2, \quad H = 2.$$

The initial conditions of the state variables were

$$(x_1, x_2) \in \{-5.5, -5.1, -4.7, 0.5, 0.9, 1.3, 1.7, 7, 7.4, 7.8\} \times \{0\} \cup \{-5, -4.6, -4.2, \ldots, 25\} \times \{-4.1\}.$$

The simulation used up $t_{CPU} = 0.94$ seconds.

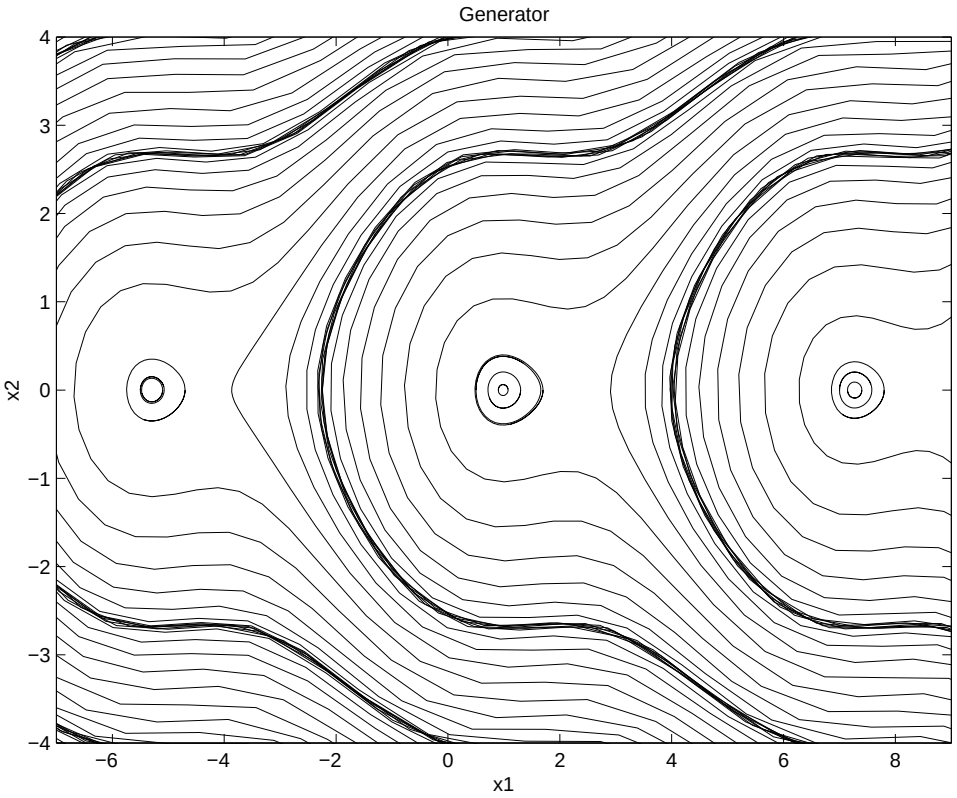


Figure 14: *Synchronous generator ($P_m = 2$, $P_e = 1.2$, $H = 1.3$)*

$$\left.\begin{array}{ll} \dot{x_1} & = x_2 \\ \dot{x_2} & = 0.85 - \sin x_1 \\ y & = 2 \sin x_1 \end{array}\right\}. \tag{51}$$

The equilibrium point of Equation 51 is

$$(1.0160 \pm 2n\pi, 0), \quad n = 0, 1, 2, \ldots,$$

and the eigenvalues at these equilibrium points are

$$-0.2997 + \jmath 0.2997, \quad \text{and} \quad 0.2997 - \jmath 0.2997.$$

The equilibrium points are unstable since there is an eigenvalue which has a real part greater than zero ([Kha96], Theorem 3.7).
From the two cases demonstrated above we can see that the simplified generator only has equilibrium points when

$$P_m \leq P_e. \tag{52}$$

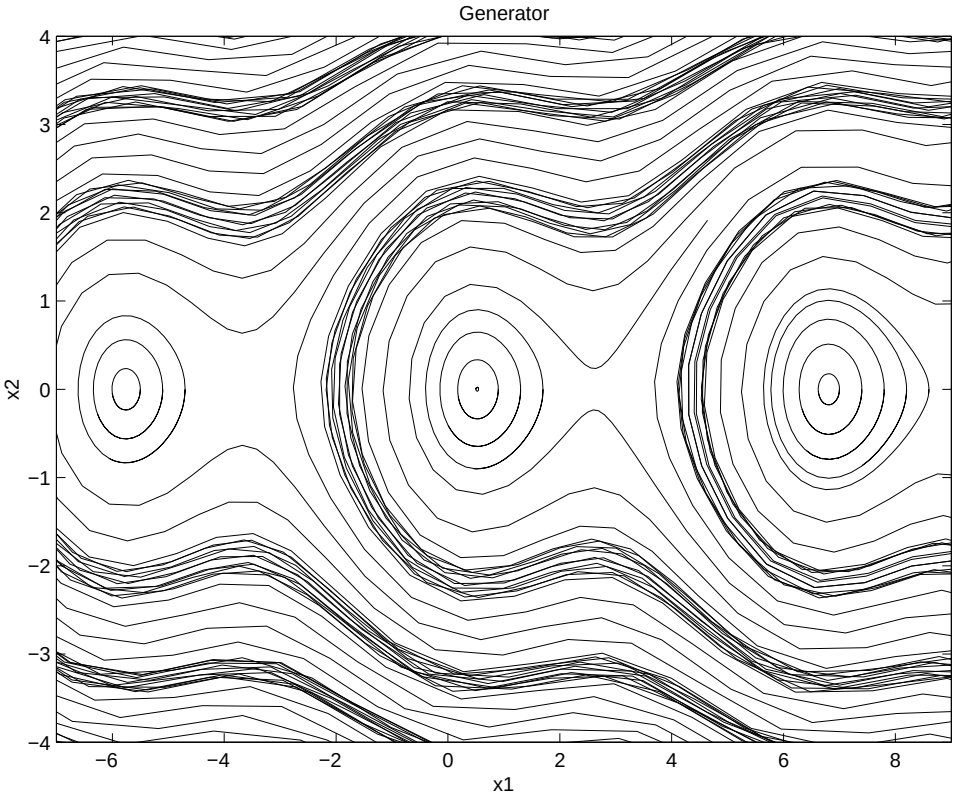


Figure 15: *Synchronous generator* $P_m = 1.7,\ P_e = 2,\ H = 2$

Domain of attraction

Write the equation again as

$$\left.\begin{array}{ll} \dot{x_1} & = x_2 \\ \dot{x_2} & = a_1 - a_2 \sin x_1 \end{array}\right\}. \tag{53}$$

Let a candidate for Lyapunov function be

$$V = \frac{1}{2}s^2, \quad s = Cx_1 + x_2,\ C > 0. \tag{54}$$

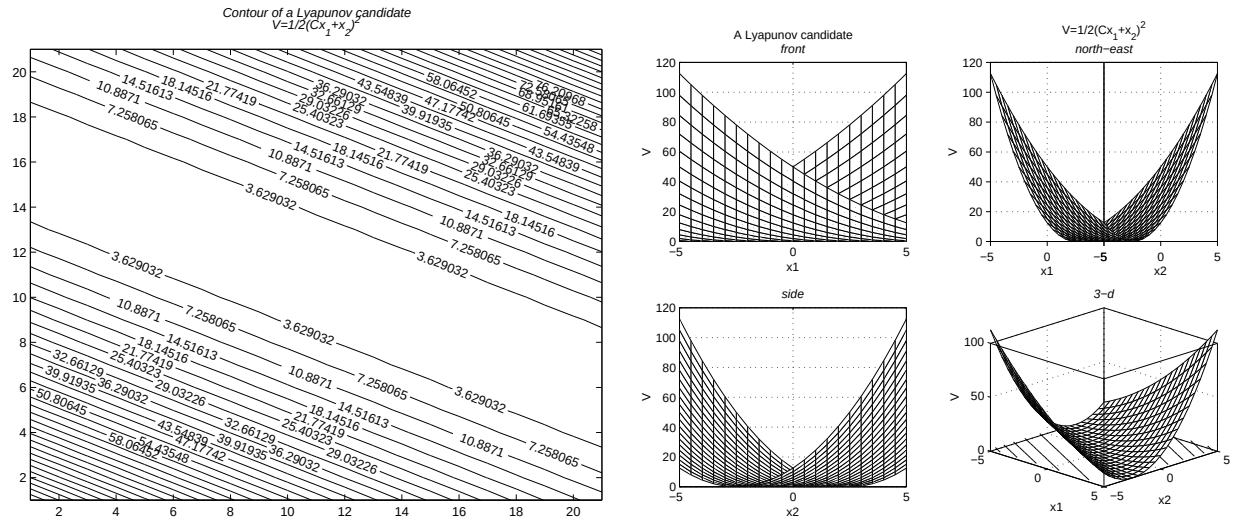
Then The contour of this Lyapunov candidate function can be described by

$$V = \frac{1}{2}\left(Cx_1 + x_2\right)^2 = \Gamma, \quad C, \Gamma > 0, \tag{55}$$

which gives

$$x_2 = \pm\sqrt{2\Gamma} - Cx_1. \tag{56}$$

Let $C = 2$,$a_1 = 1.5$, $a_2 = 1.2$ the plot and contour is shown in Figure 16 The derivative of the



(a) (b)

Figure 16: *Lyapunov candidate* $V = \frac{1}{2}\left(2x_1 + x_2\right)^2$ *The scale of axes in (a) is not the actual values of* x_1 *and* x_2

Lyapunov candidate is

$$\dot{V} = \left(Cx_1 + x_2\right)\left(C\dot{x}_1 + \dot{x}_2\right) = \left(Cx_1 + x_2\right)\left(a_1 - a_2 \sin x_1 + Cx_2\right), \tag{57}$$

and is plotted in Figure 17, from which can be seen that $\dot{V}$ is not negative-definite.

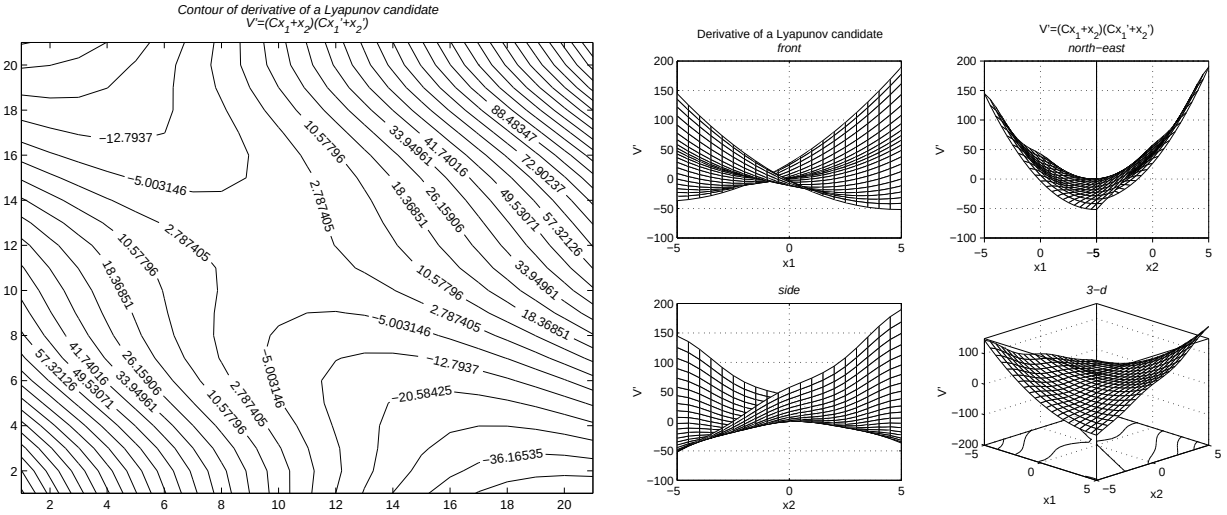


Figure 17: *Derivative of a Lyapunov candidate,* $\dot{V} = (2x_1 + x_2)\,(a_1 - a_2 \sin x_1 + 2x_2)$ *,* $a_1 = 1.5$*,* $a_2 = 1.2$ *The scale of axes in (a) does not represent the actual values of* x_1 *and* x_2

Consider another Lyapunov function

$$V = a_2 (1 - \cos x_1) + \frac{1}{2} x_2^2, \tag{58}$$

which has the derivative

$$\dot{V} \quad = a_2 \sin x_1 \dot{x}_1 + x_2 \dot{x}_2 \quad = a_2 x_2 \sin x_1 + x_2 (a_1 + a_2 \sin x_1) \quad = a_1 x_2, \tag{59}$$

from which we can see that

$$\dot{V} < 0 \quad \Longleftrightarrow \quad a_1 < 0, \quad \forall x \neq 0. \tag{60}$$

In other words this function is a successful Lyapunov candidate in the case a synchronous motor, but is not so in the case of a generator.
Consider three more Lyapunov candidates all of which are not a successful candidates, the first one is

$$\begin{aligned} V &= x_2^2 + \frac{2x_1^2}{1 + x_1^2}, & (61) \\ \dot{V} &= \frac{-4x_1 \dot{x}_1}{(1 + x_1^2)^2} + \frac{4x_1 \dot{x}_1}{1 + x_1^2} + 2x_2 \dot{x}_2, \\ &= 2 \left(a_1 - a_2 \sin x_1 - \frac{2x_1^3}{(1 + x_1^2)^2} + \frac{2x_1}{1 + x_1^2} \right) x_2, & (62) \end{aligned}$$

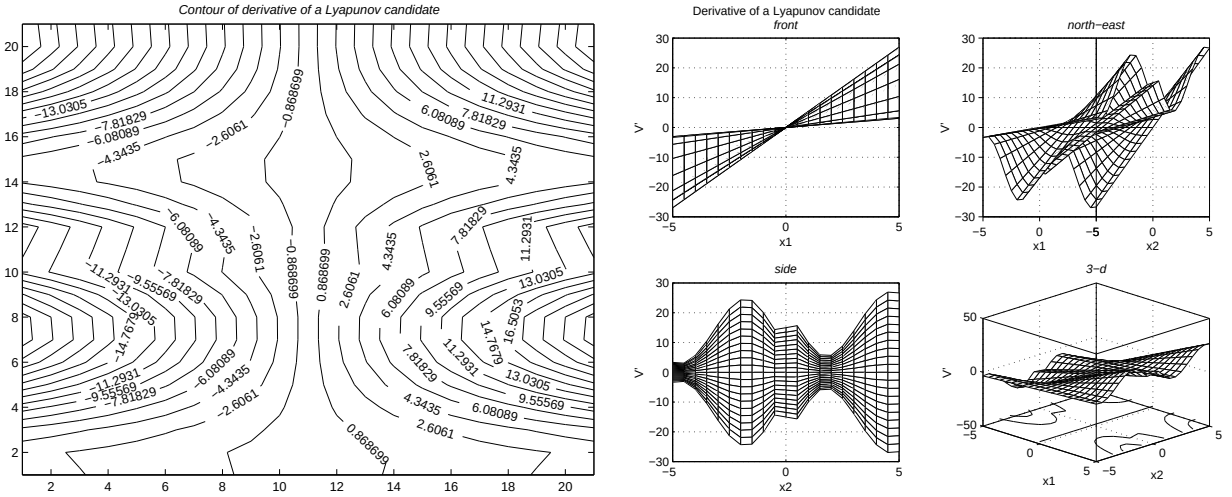
$\dot{V}$ of which is shown in Figure 18 ($a_1 = 1.5$ and $a_2 = 1.2$), the second one is

$$\begin{aligned} V &= x_1^2 + \frac{2x_2}{1 + x_2^2}, & (63) \\ \dot{V} &= 2x_1 \dot{x}_1 - \frac{4x_2 \dot{x}_2}{(1 + x_2^2)^2} + \frac{4x_2 \dot{x}_2}{1 + x_2^2}, \\ &= \frac{2x_2 \left(2 (a_1 - a_2 \sin x_1) + x_1 (1 + x_2^2)^2 \right)}{(1 + x_2^2)^2}, & (64) \end{aligned}$$

$\dot{V}$ of which is shown in Figure 19, and the third one is

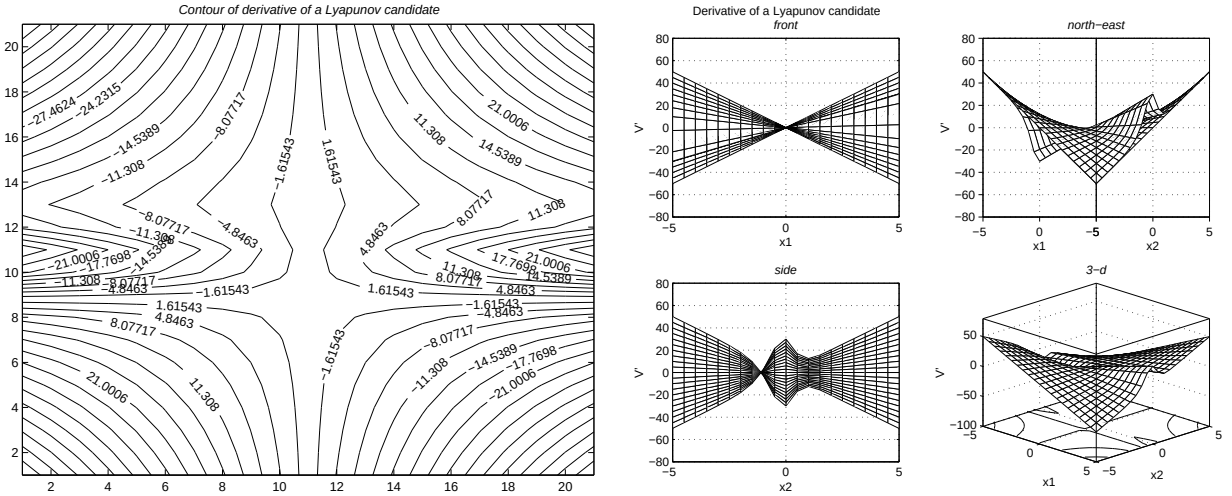
$$\begin{aligned} V &= \frac{1}{x_1^2} + \frac{1}{x_2^2}, & (65) \\ \dot{V} &= -\frac{2\dot{x}_1}{x_1^3} - \frac{2\dot{x}_2}{x_2^3}, \\ &= -\frac{2 (a_1 - a_2 \sin x_1)}{x_2^3} - \frac{2x_2}{x_1^3}, & (66) \end{aligned}$$

$\dot{V}$ of which is shown in Figure 20.



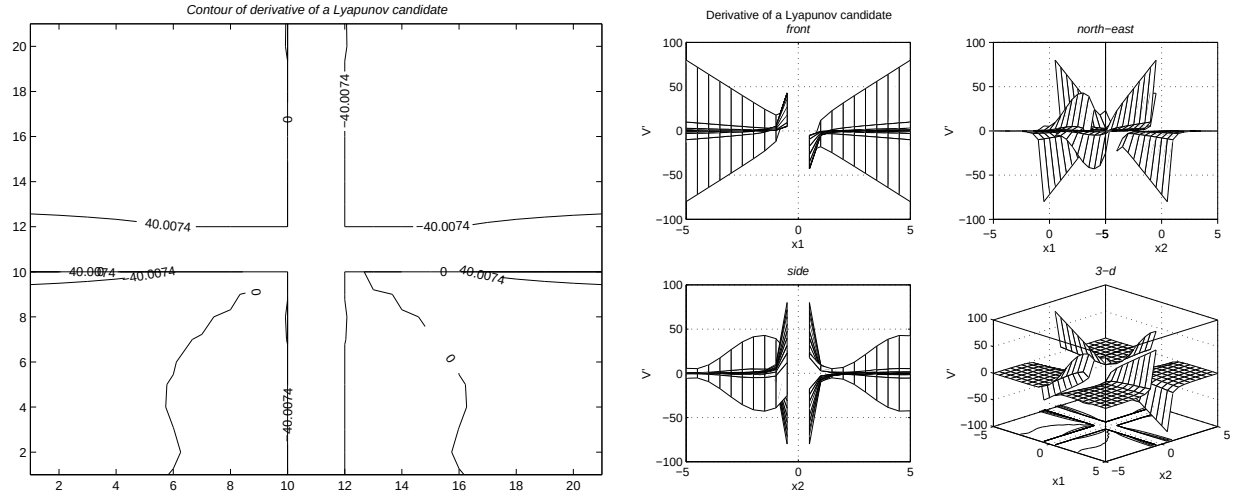
(a) (b)

Figure 18: *Derivative of a Lyapunov candidate* $\dot{V} = 2\left(a_1 - a_2 \sin x_1 - \frac{2x_1^3}{\left(1+x_1^2\right)^2} + \frac{2x_1}{1+x_1^2}\right) x_2$, $a_1 = 1.5$, $a_2 = 1.2$ *The scale of axes in (a) does not represent the actual values of* x_1 *and* x_2



(a) (b)

Figure 19: *Derivative of a Lyapunov candidate* $\dot{V} = \frac{2x_2\left(2(a_1 - a_2 \sin x_1) + x_1\left(1+x_2^2\right)^2\right)}{\left(1+x_2^2\right)^2}$, $a_1 = 1.5$, $a_2 = 1.2$ *The scale of axes in (a) does not represent the actual values of* x_1 *and* x_2



(a) (b)

Figure 20: *Derivative of a Lyapunov candidate* $\dot{V} = -\frac{2(a_1 - a_2 \sin x_1)}{x_2^3} - \frac{2x_2}{x_1^3}$ $a_1 = 1.5$, $a_2 = 1.2$ *The scale of axes in (a) does not represent the actual values of* x_1 *and* x_2

Consider another Lyapunov candidate

$$\begin{aligned} V &= \frac{1}{2}x^T Px + a_2\left(1-\cos x_1\right) && (67)\\ &= \frac{g}{l}\left(1-\cos x_1\right) + \frac{x_1}{2}\left(p_{11}x_1 + p_{12}x_2\right) + \frac{x_2}{2}\left(p_{12}x_1 + p_{22}x_2\right) \\ &= \frac{1}{2l}\left[-2g\left(-1+\cos x_1\right) + lp_{11}x_1^2 + 2lp_{12}x_1x_2 + lp_{22}x_2^2\right], && (68)\end{aligned}$$

where

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}.$$

This Lyapunov candidate will be positive definite when

$$p_{11} > 0, \quad p_{22} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0. \tag{69}$$

The derivative of the Lyapunov candidate is then

$$\begin{aligned} \dot{V} &= \frac{g}{l}\dot{x_1}\sin x_1 + \frac{\dot{x_1}}{2}\left(p_{11}x_1 + p_{12}x_2\right) + \frac{\dot{x_2}}{2}\left(p_{12}x_1 + p_{22}x_2\right) + \frac{x_1}{2}\left(p_{11}\dot{x_1} + p_{12}\dot{x_2}\right) + \frac{x_2}{2}\left(p_{12}\dot{x_1}\right) && (70)\\ &= \frac{g}{l}x_2\sin x_1 + \frac{x_2}{2}\left(p_{11}x_1 + p_{12}x_2\right) + \frac{\left(a_1 - a_2\sin x_1\right)}{2}\left(p_{12}x_1 + p_{22}x_2\right) + \\ &\quad \frac{x_1}{2}\left(p_{11}x_2 + p_{12}\left(a_1 - a_2\sin x_1\right)\right) + \frac{x_2}{2}\left(p_{12}x_2\right) \\ &= \frac{g}{l}x_2\sin x_1 + \frac{x_2}{2}\left(p_{11}x_1 + p_{12}x_2\right) + \\ &\quad \frac{1}{2}\left(a_1\left(p_{12}x_1 + p_{22}x_2\right) - a_2\left(p_{12}x_1\sin x_1 + p_{22}x_2\sin x_1\right)\right) + \\ &\quad \frac{x_1}{2}\left(p_{11}x_2 + p_{12}\left(a_1 - a_2\sin x_1\right)\right) + \frac{x_2}{2}\left(p_{12}x_2\right), && (71)\end{aligned}$$

here the terms x_1x_2 and $x_2\sin x_1$ are sign indefinite, the term $x_1\sin x_1 > 0$ for all $0 < |x_1| < \pi$. In Equation 71 the term x_1x_2 can not be cancelled out by any $p_{11} > 0$.
Let the objective of the control is the desired output

$$y_d = P_d\sin\omega t = Ha_2\sin\omega t, \tag{72}$$

therefore

$$a_2 = \text{constant}.$$

Collecting the terms and looking at coefficients lead to

$$\begin{aligned} \left(\frac{g}{l} - \frac{a_2}{2}p_{22}\right)x_2\sin x_1 = 0 \quad &\Longrightarrow \quad p_{22} = \frac{2g}{la_2} \\ p_{12}x_2^2 < 0 \quad &\Longrightarrow \quad p_{12} < 0, \end{aligned}$$

and the remaining terms which must be less than zero are

$$a_1 p_{12} x_1 + \frac{a_1 p_{22}}{2} x_2 + p_{11} x_1 x_2 - a_2 p_{12} x_1 \sin x_1.$$

An adaptive control scheme could be apply to the system by letting the Lyapunov function be

$$V = \frac{1}{2} x^T P x + a_2 \left(1 - \cos x_1\right) + \frac{1}{2} \tilde{a}_1^2. \tag{73}$$

The derivative of this Lyapunov function would then be

$$\dot{V} = \varsigma(x) + \tilde{a}_1 p_{12} x_1 + \frac{\tilde{a}_1 p_{22}}{2} x_2 + p_{11} x_1 x_2 - a_2 p_{12} x_1 \sin x_1 + \tilde{a}_1 \dot{\tilde{a}}_1, \tag{74}$$

where

$$\varsigma(x) = \left(\frac{g}{l} - \frac{a_2}{2} p_{22}\right) x_2 \sin x_1 + p_{12} x_2^2 < 0, \tag{75}$$

where $\tilde{a}_1 = a_1 - \bar{a}_1$ and $a_1 = 0$. More about adaptive control can be found in the following section.

Adaptive control

Now suppose that H and P_e are unknown and let $a_2 = \frac{1}{H}$ and $a_1 = -\frac{P_e}{H}$ so that the system equations now become

$$\begin{aligned} \dot{x_1} &= x_2 && (76)\\ \dot{x_2} &= a_1 sin x_1 + a_2 P_m + u && (77)\\ y &= -\frac{a_1}{a_2}\sin x_1 && (78) \end{aligned}$$

Let the Lyapunov function be

$$V = \frac{1}{2}s^2 + \frac{1}{2}\left[\begin{array}{cc} \tilde{a_1} & \tilde{a_2} \end{array}\right]\Gamma^{-1}\left[\begin{array}{c} \tilde{a}_1 \\ \tilde{a}_2 \end{array}\right], \tag{79}$$

where $s = C\,x_1 + x_2$ is also the hyperplane used in the control input $u = K\,\mathrm{sgn(s)}$, $K < 0$, $\tilde{a}_1 = a_1 - \bar{a}_1$ and $\tilde{a}_2 = a_2 - \bar{a}_2$. Then

$$\begin{aligned} \dot{V} &= C^2 x_1 x_2 + C\,x_2^2 + a_1\,(Cx_1 + x_2)\sin(x_1) + a_2 P_m\,(Cx_1 + x_2) + \\ &\quad K\,(C\,x_1 + x_2)\,\mathrm{sgn}\,(\mathrm{C\,x_1 + x_2}) + \left[\begin{array}{cc} \tilde{a_1} & \tilde{a_2} \end{array}\right]\Gamma^{-1}\left[\begin{array}{c} \dot{\tilde{a}}_1 \\ \dot{\tilde{a}}_2 \end{array}\right] && (80)\\ &= C^2 x_1 x_2 + C\,x_2^2 + \bar{a}_1\,(Cx_1 + x_2)\sin(x_1) + \bar{a}_2 P_m\,(Cx_1 + x_2) + \\ &\quad K\,|C\,x_1 + x_2| + \left[\begin{array}{cc} \tilde{a_1} & \tilde{a_2} \end{array}\right]\left[\begin{array}{c} (C\,x_1 + x_2)\sin(x_1) \\ (C\,x_1 + x_2)\,P_m \end{array}\right] + \left[\begin{array}{cc} \tilde{a_1} & \tilde{a_2} \end{array}\right]\Gamma^{-1}\left[\begin{array}{c} \dot{\tilde{a}}_1 \\ \dot{\tilde{a}}_2 \end{array}\right] && (81) \end{aligned}$$

From this we derive the equations for adaptivity which are

$$\left[\begin{array}{c} \tilde{a}_1 \\ \tilde{a}_2 \end{array}\right] = -\Gamma\left[\begin{array}{c} (C\,x_1 + x_2)\sin(x_1) \\ (C\,x_1 + x_2)\,P_m \end{array}\right], \tag{82}$$

which, when

$$-\Gamma = \left[\begin{array}{cc} \gamma_1 & 0 \\ 0 & \gamma_2 \end{array}\right] \tag{83}$$

and

$$a_1 = a_2 = 0 \tag{84}$$

gives

$$\begin{aligned} \dot{\tilde{a}}_1 &= \gamma_1\,(C\,x_1 + x_2)\sin(x_1) && (85)\\ \dot{\tilde{a}}_2 &= \gamma_2\,(C\,x_1 + x_2)\,P_m. && (86) \end{aligned}$$

Let us look at the effect of the signum function in the input by letting

$$u = K\,\mathrm{sgn}\,(\mathrm{C\,x_1 + x_2})\,. \tag{87}$$

The result is shown in Figure 21. The system simulated in Figure 21 had

Figure 21 (a) and (b), $P_m = -2,\ P_e = 1.2,\ H = 1.3,\ K = -3,\ C = 3,$

Figure 21 (c), $P_m = 2,\ P_e = 1.2,\ H = 1.3,\ K = -3,\ C = 3.$ (notice that the sign of P_m are different)

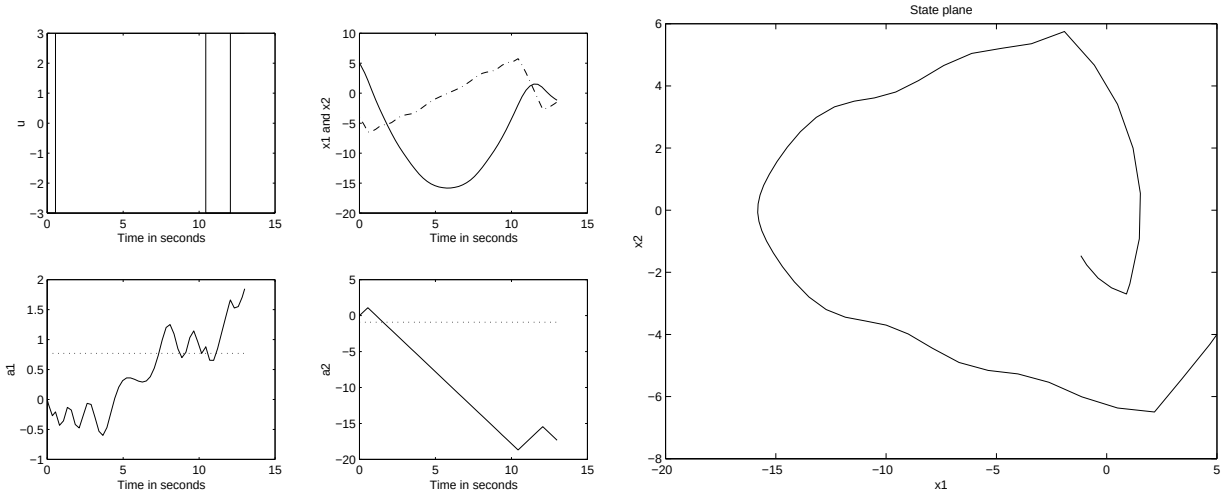
The initial conditions for Figure 21 (a) and (b) were

$$(x_1, x_2) = (5, -5),$$

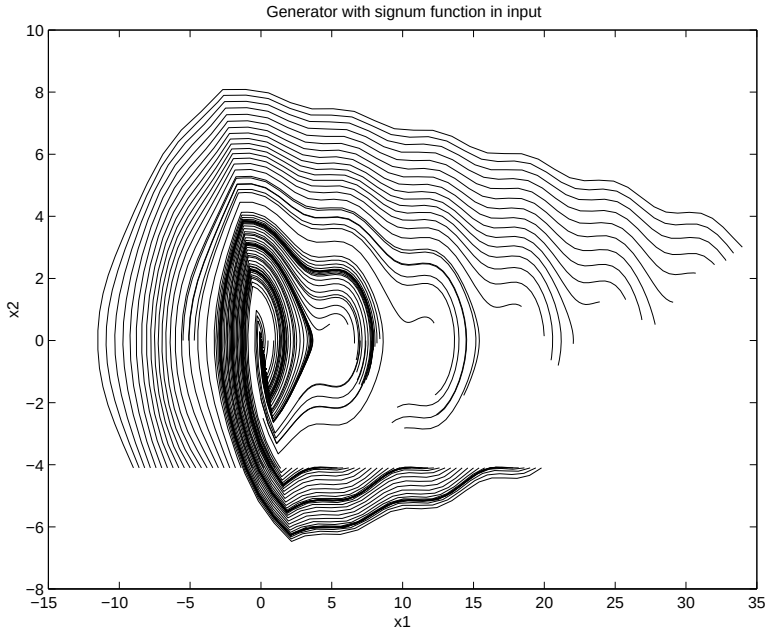
while those for Figure 21 (c) were

$$(x_1, x_2) \in \{-5.5, -5.1, -4.7, 0.5, 0.9, 1.3, 1.7, 7, 7.4, 7.8\} \times \{0\} \cup \{-9, -8.6, -8.2, \ldots, 19.8\} \times \{-4.1\}\,.$$

Miscellaneous details from the simulation were, for (a) and (b), $t_{CPU} = 0.0200\,\text{sec}$, $t_{simulation} = 13\,\text{sec}$, and file size of (a)= 12.50 KB,of (b)= 7.07 KB, and for (c) total CPU time $t_{CPU} = 36.32\,\text{sec}$, file size = 867.87 KB.



(a) (b)



(c)

Figure 21: *Synchronous generator with signum function in input* $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$. *(a) time domain, (b) state space* ($P_m = -2$ *for (a) and (b)), and (c) state space* ($P_m = 2$ *for (c)).*

Next use the adaptive control scheme

$$u = -P_e x_1 - \bar{a}_1 \sin x_1 + K \operatorname{sgn}(\mathrm{C\,x_1 + x_2}) \tag{88}$$

to obtain the result as shown from Figure 22 to Figure 26. The conditions of the simulation were

$$P_m = 2, \quad P_e = 1.2, \quad H = 1.3, \quad K = -3, \quad C = 3, \quad \gamma_1 = \gamma_2 = -5,$$

The initial conditions were

Figure 22 $(x_1, x_2) = (5, -5)$,

Figure 23 $(x_1, x_2) = (5, -5)$,

Figure 24 $(x_1, x_2) \in \{-5.5, -5.1, -4.7, 0.5, 0.9, 1.3, 1.7, 7, 7.4, 7.8\} \times \{0\} \cup \{-9, -8.6, -8.2, \ldots, 19.8\} \times \{-4.1\}$ (planned),

Figure 25 $(x_1, x_2) = (3, 3), (-3, -3), (3, -3)$, and

Figure 26 (a) $(x_1, x_2) = (\varsigma, \varsigma), (\varsigma, -\varsigma)$, (b) $(x_1, x_2) = (\pm 5, \varsigma), (\varsigma, \pm 5)$, $\varsigma = (-5, -3.3, -1.6, 0.1, 1.8, 3.5)$.

Miscellaneous informations related to the simulation were

Figure 22 $t_{CPU} = 0.0400\,\mathrm{sec}$, $t_{simulation} = 10\,\mathrm{sec}$, file size = 13.31 KB,

Figure 23 $t_{CPU} = 0.0400\,\mathrm{sec}$, $t_{simulation} = 10\,\mathrm{sec}$, file size = 7.07 KB,

Figure 24 $t_{CPU} = 14.5600\,\mathrm{sec}$, file size = 13.31 KB,

Figure 25 file size of (a) = 42.99 KB, file size of (b) = 10.87 KB,

Figure 26 (a) file size = 47.83 KB, (b) file size = 90.68 KB.

Figure 25 and 26 are similar to Figure 22 to 24 except that $t_{simulation} = 30$ seconds instead of $t_{simulation} = 10$ seconds.
Figure 22, the input u seems to be getting larger and larger, may be running away. Initial conditions $x_1(0) > 0$ therefore a generator, $x_2(0) < 0$ therefore $\dot{\delta} < \omega_R$. After approx. 1 second the machine actually becomes a motor instead of a generator ($x_1 < 0$), which may be the result of injecting a signum function into the system. The estimates $\bar{a}_1(t)$ and $\bar{a}_2(t)$ did not converge to the true values. The plot for $\bar{a}_2(t)$ was unnecessary since it was never used in the control equation.
Figure 23, the states plot did go around in circles but since the circulation radius seems to be getting larger and larger the system is still unstable.
Figure 24, again the states plot shows a considerable degree of unstability because the trajectory gradually expands. Both Figure 24 and 23 does show that the controlling action makes the system behave better than when running without control.

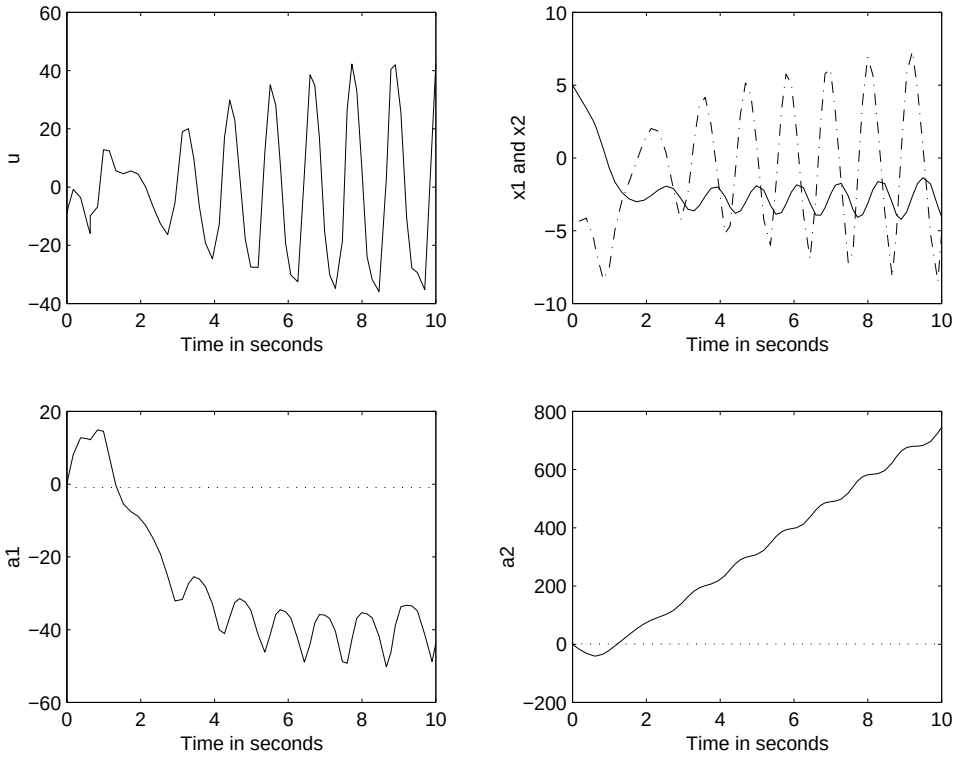


Figure 22: *Synchronous generator with adaptive control* $P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$.

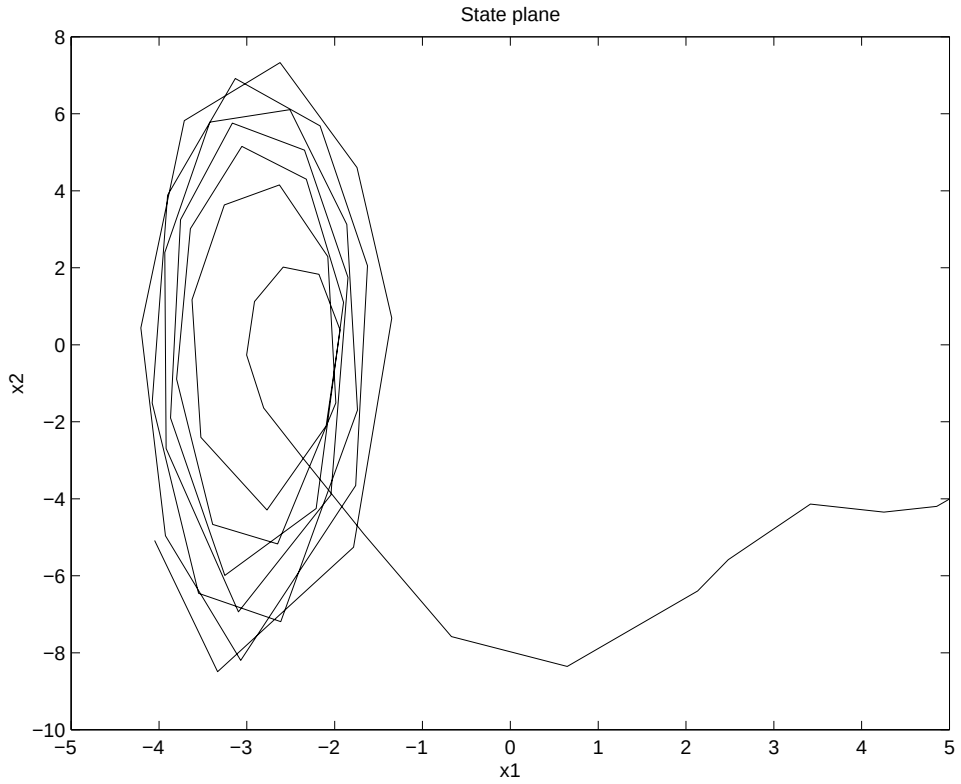


Figure 23: *Synchronous generator with adaptive control* ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$

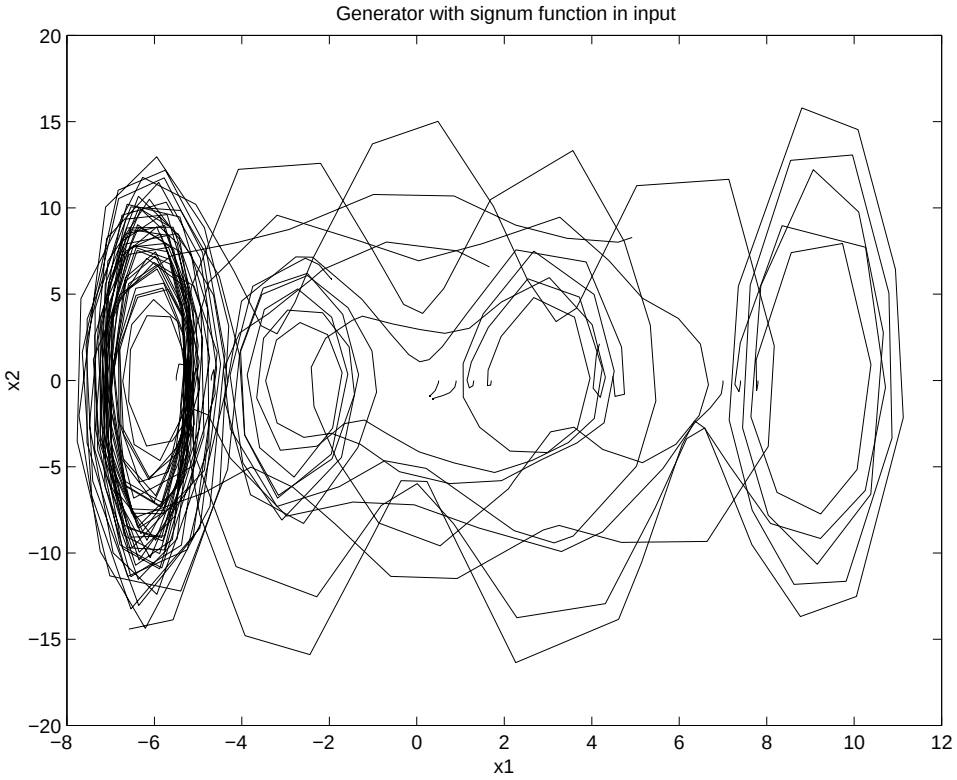
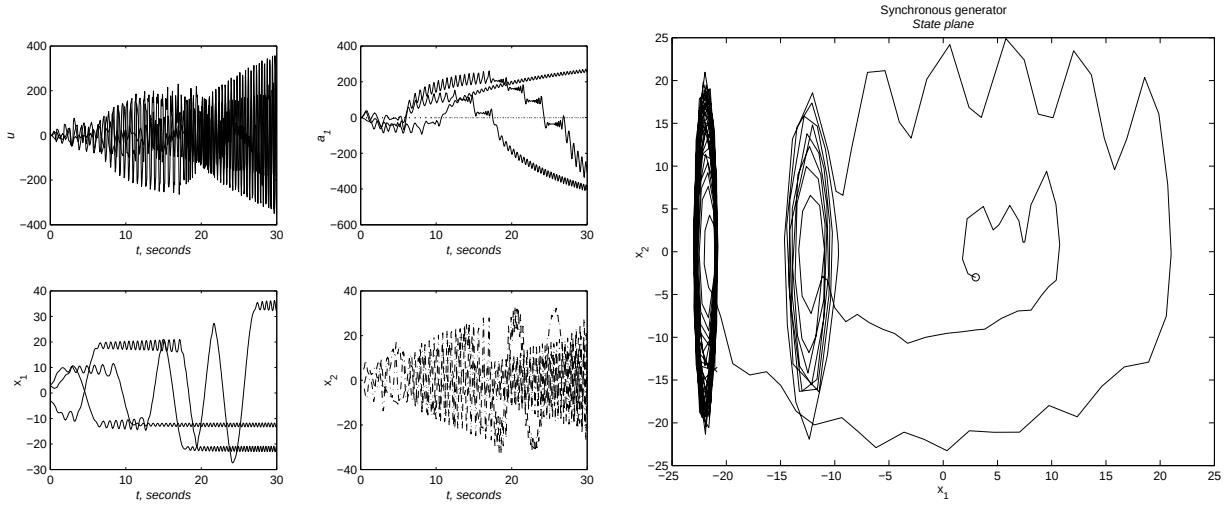
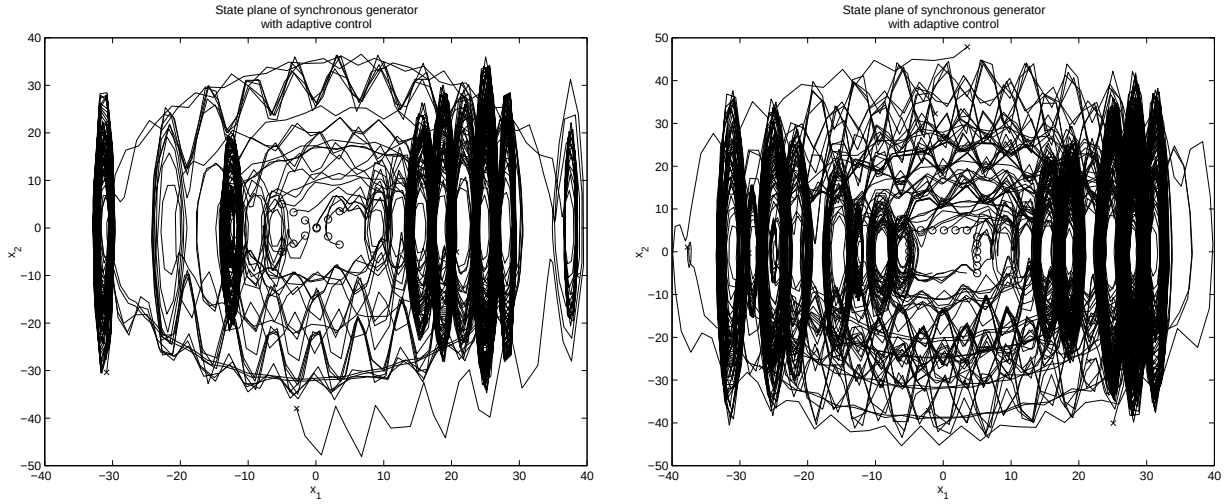


Figure 24: *Synchronous generator with adaptive control* $P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$



(a) (b)

Figure 25: *Synchronous generator with adaptive control* $P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$



(a) (b)

Figure 26: *Synchronous generator with adaptive control* $P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$

Use an adaptive control with

$$u = -P_e x_1 - \bar{a}_1 \sin x_1 + K \operatorname{sgn}(\mathrm{C\,x_1 + x_2}) + \mathrm{v}, \tag{89}$$

where v is a dither signal. The result when $\sigma_v^2 = 0.5$ is shown in Figure 27. The result shows an improvement in parameter estimation.

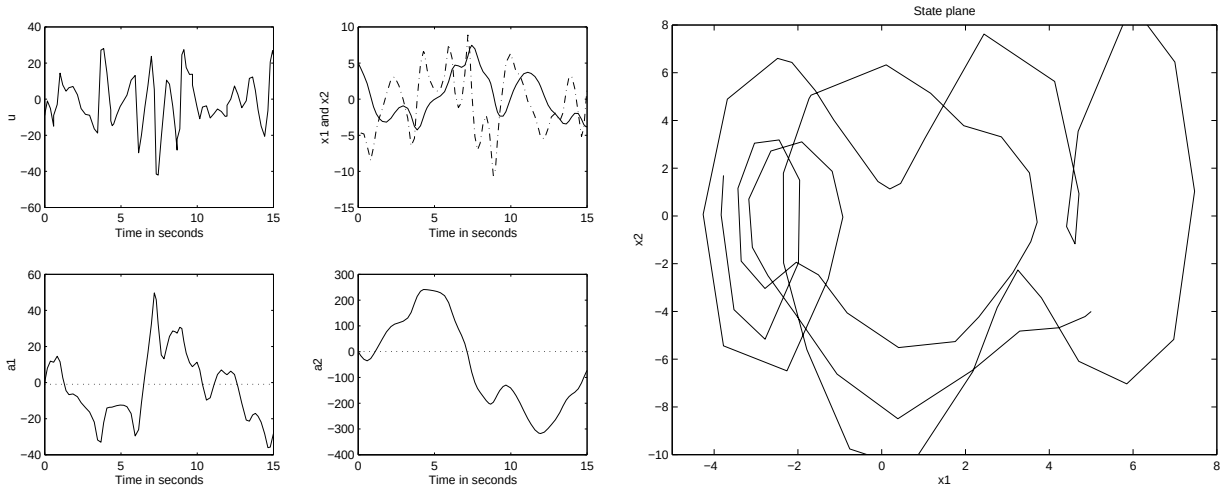


Figure 27: *Synchronous generator with adaptive control with additional random signal in input ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$, $\sigma_v^2 = 0.5$. (a) time domains, (b) state plane*

Figure 27 has the initial conditions $(x_1, x_2) = (5, -5)$. Miscellaneous details of the simulations were for (a) $t_{CPU} = 0.0500\,\mathrm{sec}$, $t_{simulation} = 15\,\mathrm{sec}$, file size = 13.71 KB, and for (b) $t_{CPU} = 0.0500\,\mathrm{sec}$, $t_{simulation} = 15\,\mathrm{sec}$, file size = 7.51 KB.
But increased the simulation time from 15 to 40 seconds and found out the inconsistency of the parameter estimates (Figure 28).
Figure 31 has $\sigma_v^2 = 0.5$. The initial conditions were $(x_1, x_2) = (3, 3), (-3, -3), (3, -3)$ for (a) and (b), $(x_1, x_2) = (\varsigma, \varsigma), (\varsigma, -\varsigma)$ for (d) and $(x_1, x_2) = (\pm 5, \varsigma), (\varsigma, \pm 5)$ for (e), $\varsigma = (-5, -3.3, -1.6, 0.1, 1.8, 3.5$
The files' sizes were for (a) 67.06 KB, for (b) 20.10 KB, for (c) 23.59 KB, for (d) 52.60 KB, and for (e) 91.44 KB. In Figure 31 one can see that at the point where the parameter estimate $\bar{a}_1$ went wrong. At that point the input u also increased, the behaviour of the state variables x_1 and x_2 changed.

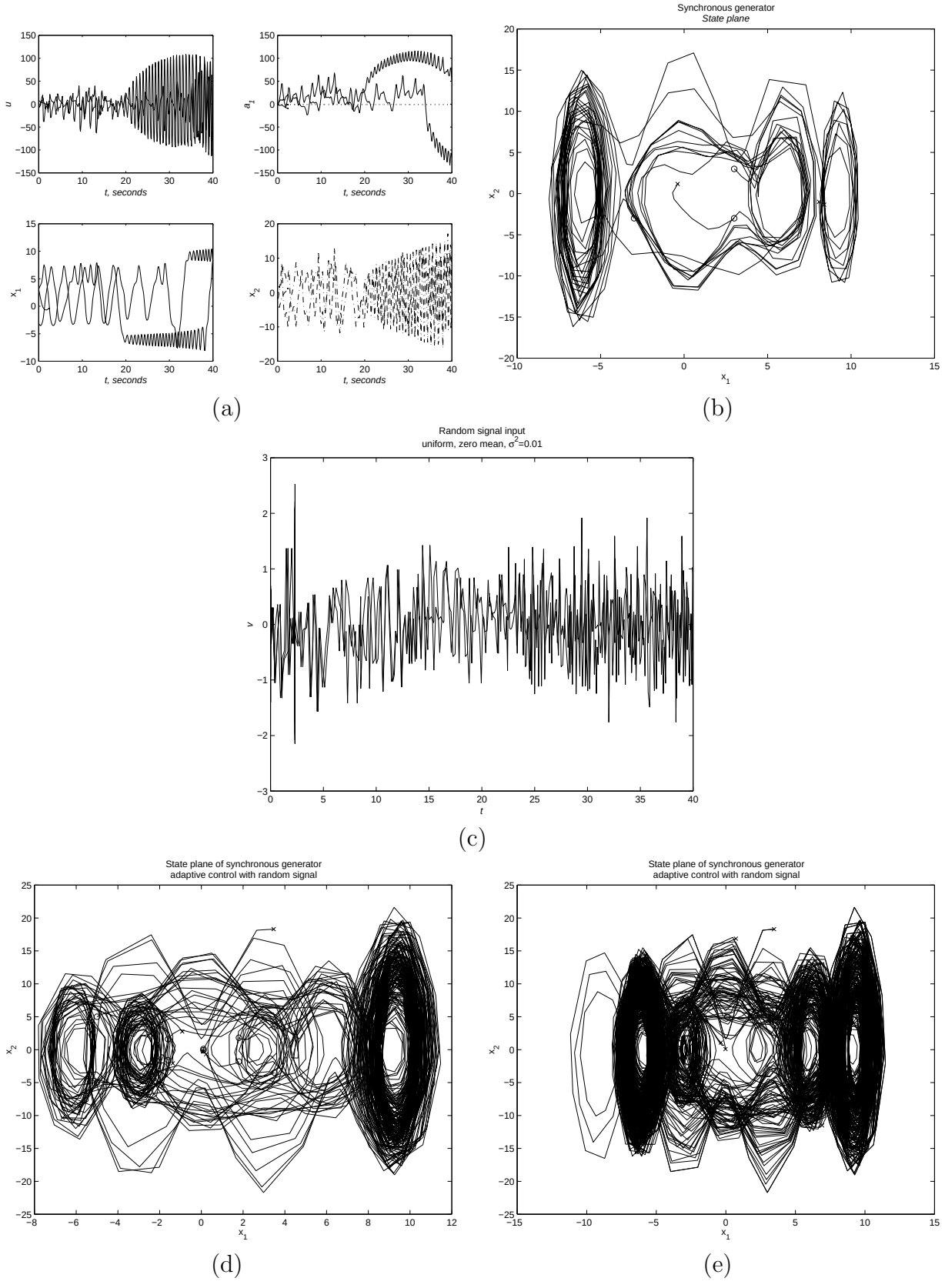


Figure 28: *Synchronous generator with adaptive control with additional random signal in input ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$, $\sigma_v^2 = 0.5$. (a) $u(t)$, $a_1(t)$, $x_1(t)$, and $x_2(t)$, (b) state plane, (c) random signal input, (d) and (e) state planes.*

Figure 29 shows the simulation result when $\sigma_v^2 = 0.9$.

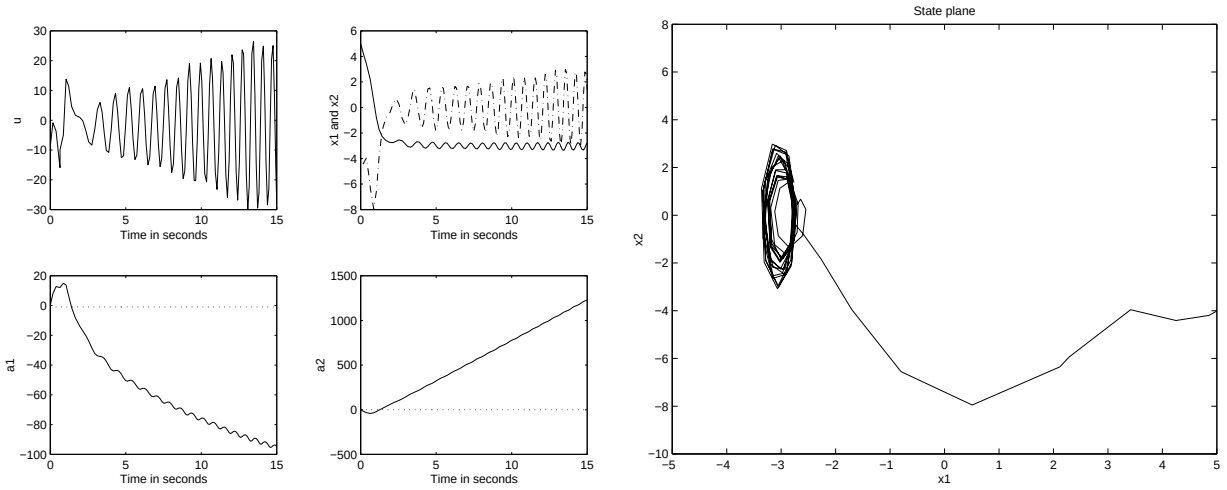


Figure 29: *Synchronous generator with adaptive control with additional random signal in input* $P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$, $\sigma_v^2 = 0.9$

Figure 29 shows the result when $\sigma_v^2 = 0.9$. The initial conditions were for (a) $(x_1, x_2) = (5, -5)$, and for (b) $(x_1, x_2) = (5, -5)$. Miscellaneous details for (a) are $t_{CPU} = 0.0700\,\text{sec}$, $t_{simulation} = 15\,\text{sec}$, file size = 15.94 KB, and for (b) $t_{CPU} = 0.0500\,\text{sec}$, $t_{simulation} = 15\,\text{sec}$, file size = 8.16 KB. The system seemed to be unstable since upto the time $t = 15$ seconds both the input u and the state variable x_2 seemed to keep growing larger. Also, the estimates for the parameters a_1 and a_2 did not converge.

The result when $\sigma_v^2 = 100$ is shown in Figure 30.

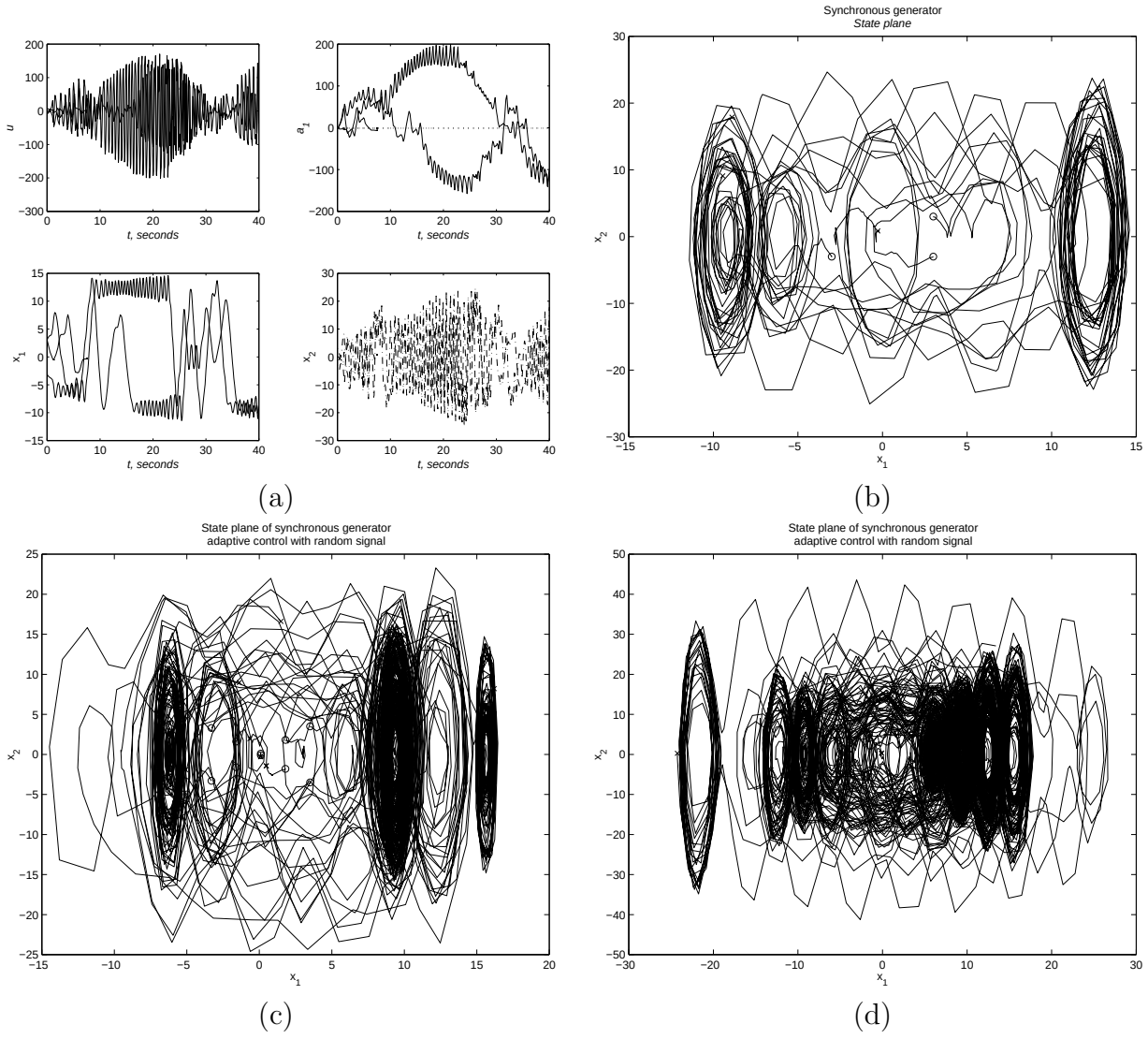


Figure 30: *Synchronous generator with adaptive control ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$, $\sigma_v^2 = 100$, (a) $u(t)$, $a_1(t)$, $x_1(t)$, and $x_2(t)$, (b), (c), and (d) state plane*

Figure 30 shows the case when $\sigma_v^2 = 100$, the initial conditions were $(x_1, x_2) = (3, 3), (-3, -3), (3, -3)$ for Figure 30 (a) and (b), for (c) $(x_1, x_2) = (\varsigma, \varsigma), (\varsigma, -\varsigma)$, and for (d) $(x_1, x_2) = (\pm 5, \varsigma), (\varsigma, \pm 5)$, $\varsigma = (-5, -3.3, -1.6, 0.1, 1.8, 3.5)$. The size of the file of Figure 30 (a) is 137.81 KB, (b) 34.68 KB, (c) 108.92 KB, and (d) 117.50 KB.

The result when $\sigma_v^2 = 0.01$ is shown in Figure 31.
Figure 31 is when $\sigma_v^2 = 0.01$. The initial conditions were $(x_1, x_2) = (3,3), (-3,-3), (3,-3)$ for (a) and (b), $(x_1, x_2) = (\varsigma, \varsigma), (\varsigma, -\varsigma)$ for (d) and $(x_1, x_2) = (\pm 5, \varsigma), (\varsigma, \pm 5)$ for (e), $\varsigma = (-5, -3.3, -1.6, 0.1, 1.8, 3.5)$. The files' sizes for (a) was 196.47 KB, for (b) 44.37 KB, for (c) 63.77 KB, for (d) 53.30 KB, and for (e) 100.37 KB. Looking at Figure 31 in comparison with Figure 30 one can see that the additional random signal introduced was unnecessary since the behaviour of the state trajectory are similar for both eventhough their σ^2 are very much different.

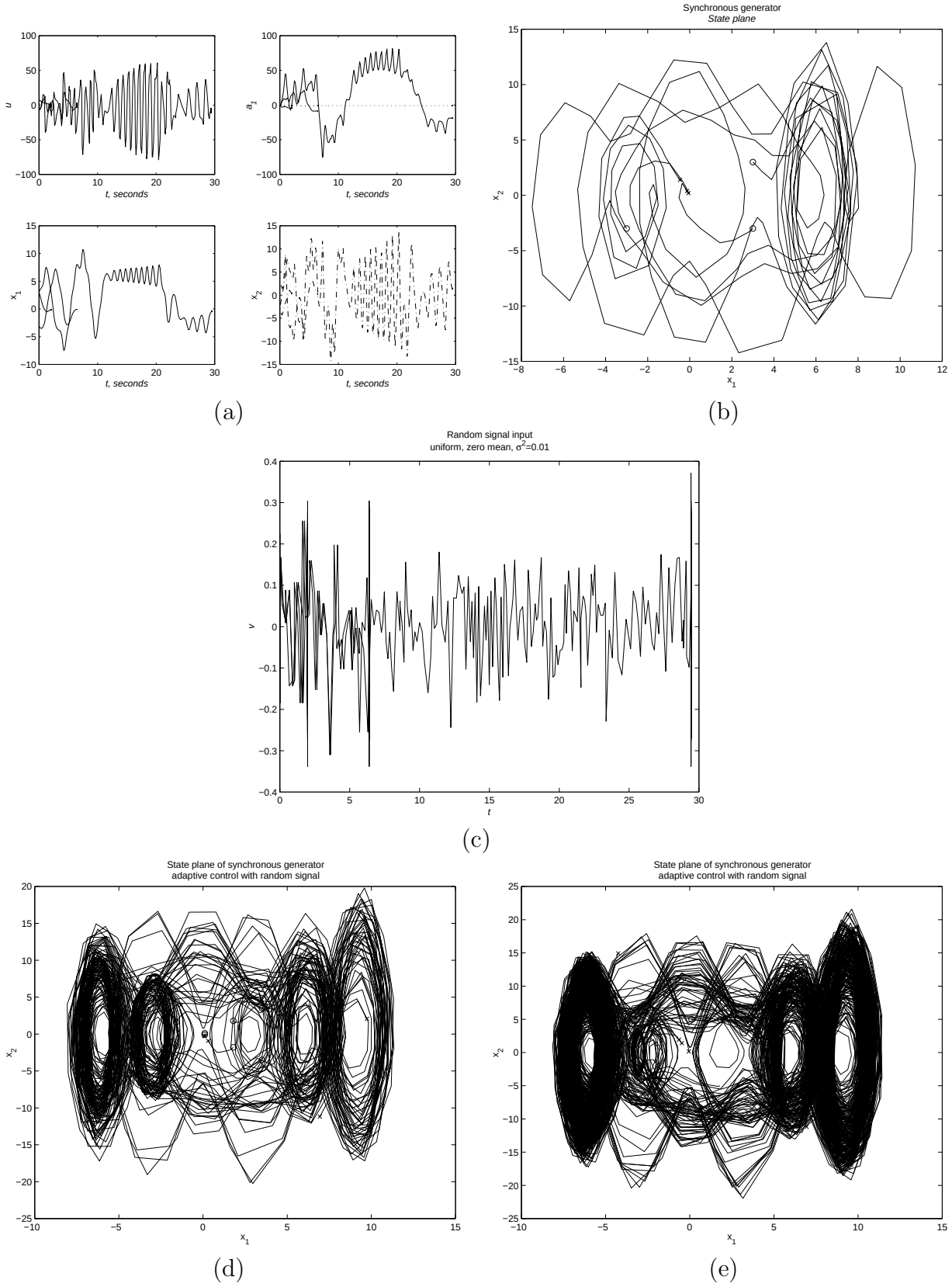


Figure 31: *Synchronous generator with adaptive control ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$, $\sigma_v^2 = 0.01$, (a) $u(t)$, $a_1(t)$, $x_1(t)$, and $x_2(t)$, (b) state plane, (c) random signal input, (d) and (e) state planes.*

Next used the input

$$u = -P_e x_1 - \bar{a}_1 \sin x_1 + K \operatorname{sgn}\left(\mathrm{C\,x_1 + x_2 e^{-\alpha t} sgn}\left(\mathrm{x_1^2 + x_2^2 - r^2}\right)\right) + \mathrm{v}. \tag{90}$$

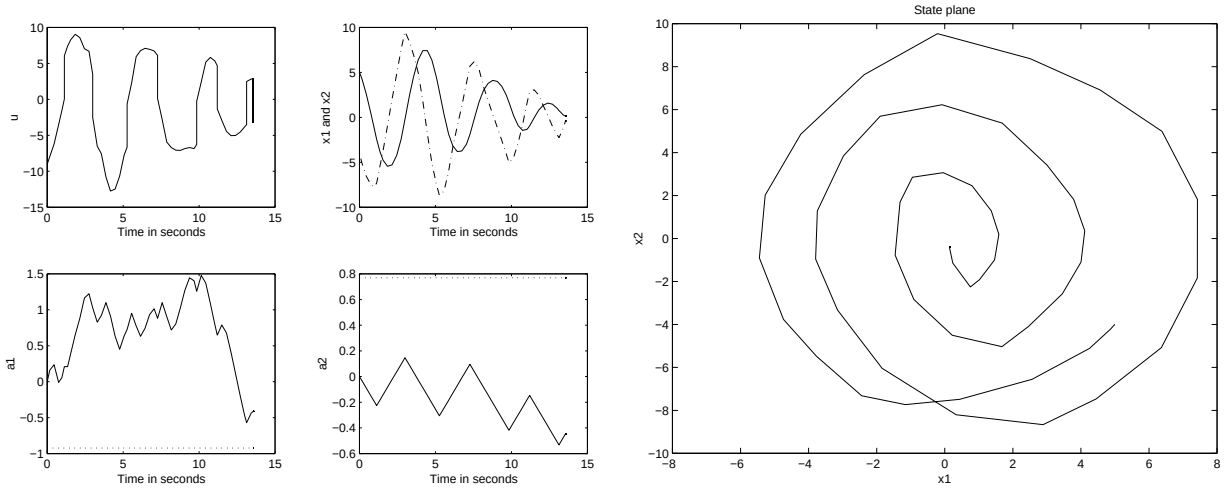
With the condition

$$P_m = 2, P_e = 1.2, H = 1.3, K = -3, C = 3, \gamma_1 = -1, \gamma_2 = -0.1, \sigma_v^2 = 0.5, \alpha = -0.8, r^2 = 1000,$$

and the initial condition

$$(x_1, x_2) = (5, -5)$$

obtained the result shown in Figure 32.



(a) (b)

Figure 32: *Synchronous generator with adaptive control with additional random signal ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = -1$, $\gamma_2 = -0.1$, $\sigma_v^2 = 0.5$, $\alpha = -0.8$, $r^2 = 1000$. (a) u, x_1, x_2, a_1, a_2, (b) state plane.*

Miscellaneous details of the simulation, for Figure 32 $t_{CPU} = 12.6100\,\mathrm{sec}$, $t_{simulation} = 15\,\mathrm{sec}$. Figure 32, (a) file size = 260.10 KB, (b) file size = 40.51 KB. Figure 32 shows both x_1 and x_2 to be approaching zero, which is not a good response for a synchronous generator.

Again do the same as above with a different set of parameters (Figure 33). The parameters were

$$P_m = 2, P_e = 1.2, H = 1.3, K = -3, C = 3, \gamma_1 = gamma_2 = 0.5, \sigma_v^2 = 0.02, \alpha = -2, r^2 = 10^5,$$

and an intial condition

$$(x_1, x_2) = (5, -5).$$

The result when simulated without the random signal was very similar to these two pictures, therefore is not shown here. Again this means that the random signal added with the hope that it would help making better estimates $\bar{a}_1$, was not successful.

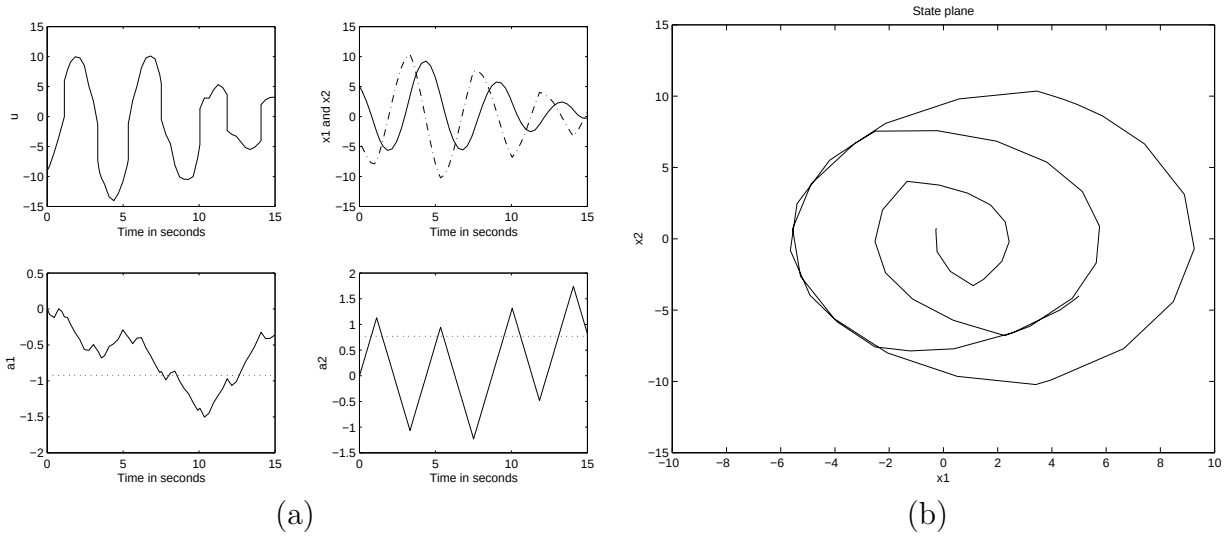


Figure 33: *Synchronous generator with adaptive control with an addition random signal ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = gamma_2 = 0.5$, $\sigma_v^2 = 0.02$, $\alpha = -2$, $r^2 = 10^5$, (a) u, x_1, x_2, a_1, a_2, (b) state plane.*

Miscellaneous details were $t_{CPU} = 0.1100\,\text{sec}$, $t_{simulation} = 15\,\text{sec}$, for (a) the file size = 13.13 KB, and for (b) 7.42 KB. Figure 33

When the σ_v^2 is increased to 30 the result is shown in Figure 34. The initial conditions were

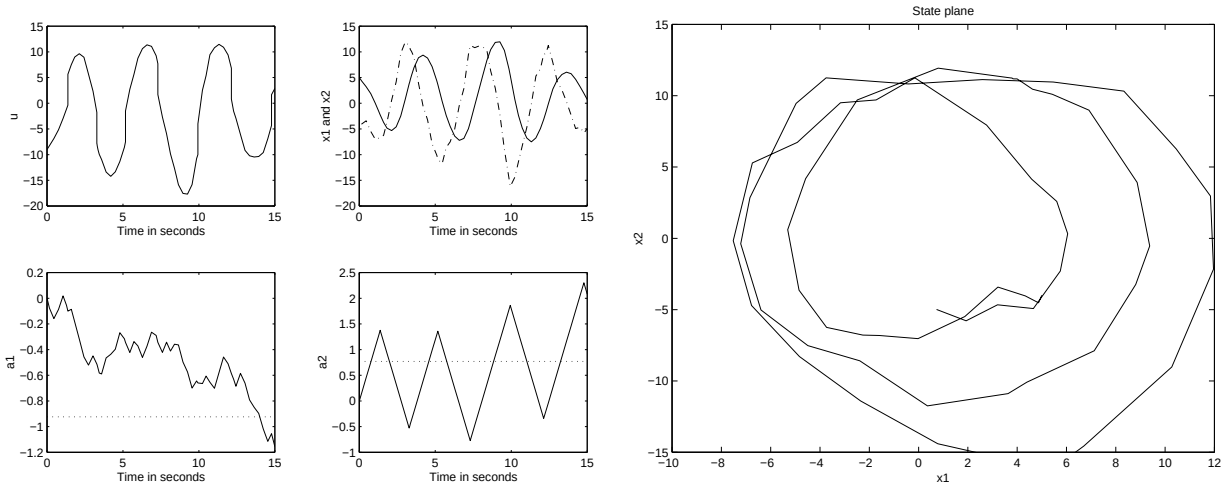
$$(x_1, x_2) = (5, -5),$$

whie other conditions were

$$P_m = 2, P_e = 1.2, H = 1.3, K = -3, C = 3, \gamma_1 = gamma_2 = 0.5, \sigma_v^2 = 30, \alpha = -2, r^2 = 10^5.$$

Miscellaneous details were

$$t_{CPU} = 0.1500\,\text{sec}, t_{simulation} = 15\,\text{sec}, filesize for (a) is 13.76KB, for (b) 7.54KB.$$



(a) (b)

Figure 34: *Synchronous generator with adaptive control with an additional uniform random signal* ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = gamma_2 = 0.5$, $\sigma_v^2 = 30$, $\alpha = -2$, $r^2 = 10^5$

The input which was tried next was

$$u = -P_e x_1 + \bar{a}_1 \sin x_1 + \bar{a}_2 P_m + K \operatorname{sgn}\left(\mathrm{C\,x_1} + \mathrm{x_2 e^{-\alpha t}} \operatorname{sgn}\left(\mathrm{x_1^2 + x_2^2 - r^2}\right)\right), \tag{91}$$

where $\bar{a}_1$ and $\bar{a}_2$ was described in Equation 85 and 86 respectively. With

$$\gamma_1 = \gamma_2 = 5, \quad \alpha = -3.2, \quad r^2 = 5 \times 10^5,$$

the result was shown in Figure 35.
Figure 35 has got initial conditions $(x_1, x_2) = (3,3), (-3,-3), (3,-3)$ for (a) and (b), $(x_1, x_2) = (\varsigma, \varsigma), (\varsigma, -\varsigma)$ for (c), and $(x_1, x_2) = (\pm 5, \varsigma), (\varsigma, \pm 5)$ for (d), where $\varsigma = (-5, -3.3, -1.6, 0.1, 1.8, 3.5)$. The file sizes were 48.56 KB for (a), 9.46 KB for (b), 13.09 KB for (c), and 16.99 KB for (d).

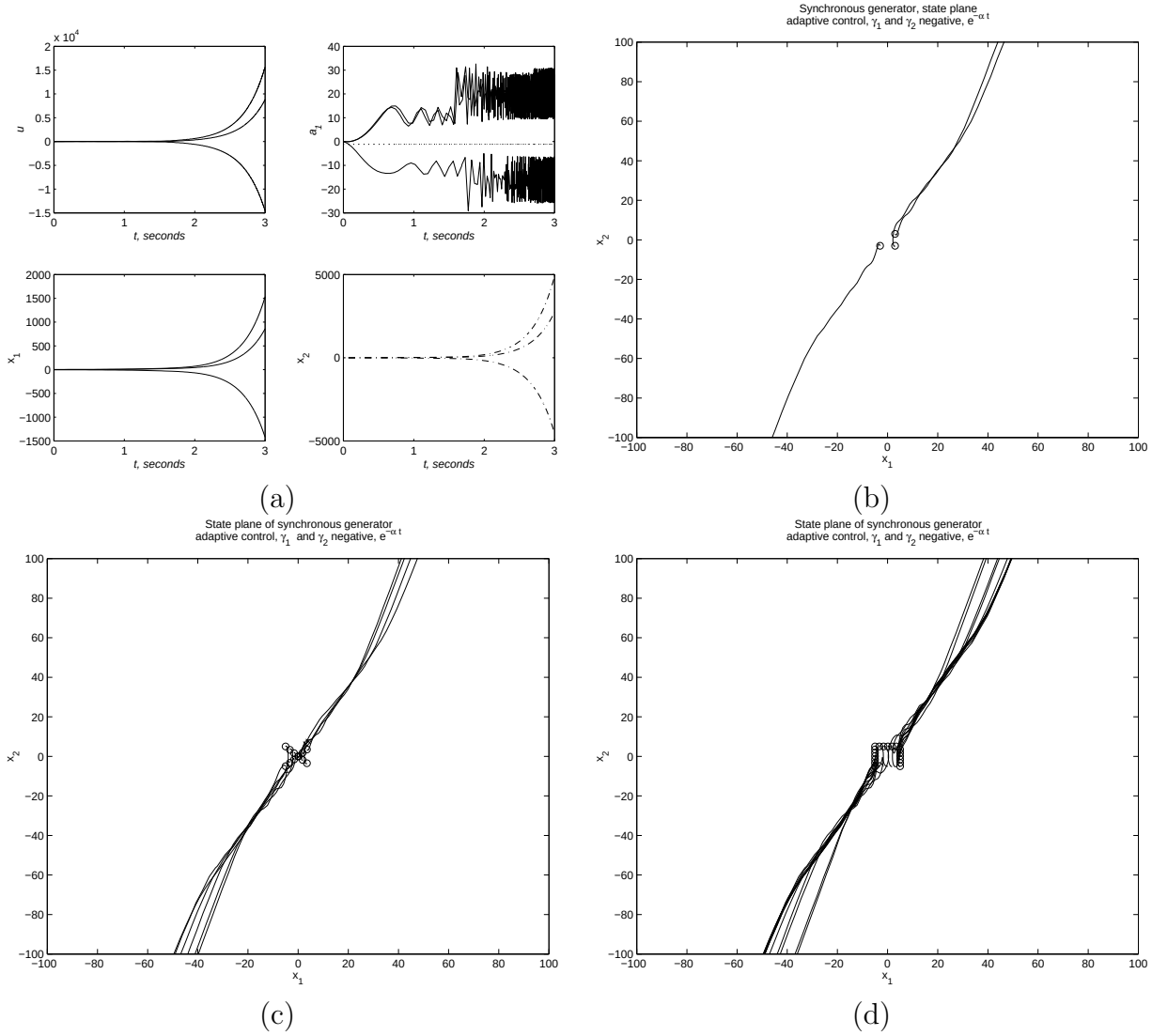


Figure 35: *Synchronous generator with adaptive control* ($P_m = 1.1$, $P_e = 1.5$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = 5$, $\alpha = -3.2$, $r^2 = 5 \times 10^5$, *(a)* $u(t)$, *(b)* $\bar{a}_1(t)$, *(c)* $x_1(t)$, *(d)*$x_2(t)$

With

$$\gamma_1 = \gamma_2 = -5, \quad \alpha = -3.2, \quad r^2 = 100,$$

the result was shown in Figure 36.
Figure 36 has got initial conditions $(x_1, x_2) = (3, 3), (-3, -3), (3, -3)$ for (a) and (b), $(x_1, x_2) = (\varsigma, \varsigma), (\varsigma, -\varsigma)$ for (d), and $(x_1, x_2) = (\pm 5, \varsigma), (\varsigma, \pm 5)$ for (e), where $\varsigma = (-5, -3.3, -1.6, 0.1, 1.8, 3.5)$. The file sizes were 32.68 KB for (a), 12.98 KB for (b), 11.29 KB for (c), 22.87 KB for (d), and 49.84 KB for (e).

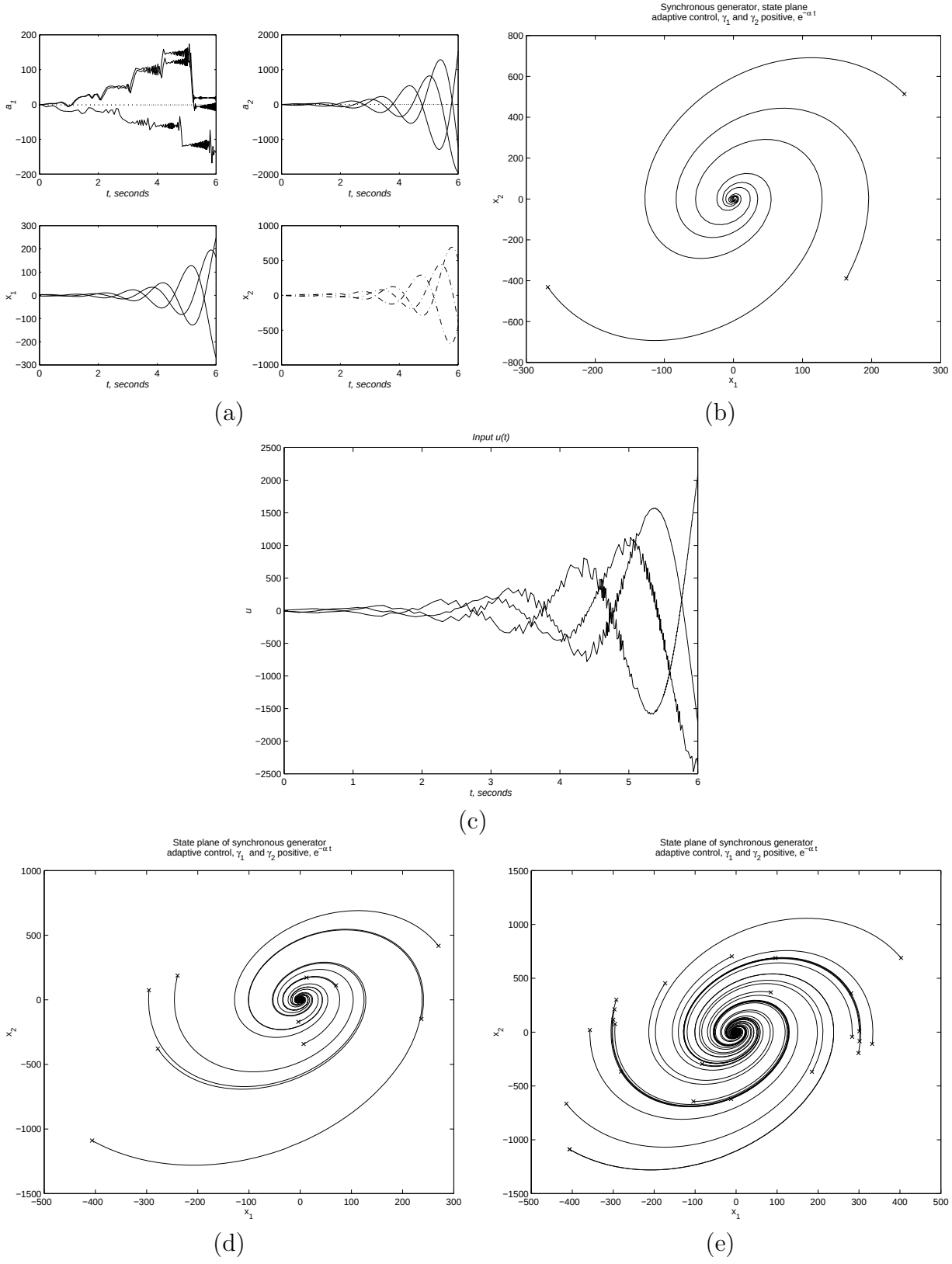


Figure 36: *Synchronous generator with adaptive control* ($P_m = 1.1$, $P_e = 1.5$, $H = 1.3$, $K = -3$, $C = 3$, $\gamma_1 = \gamma_2 = -5$, $\alpha = -3.2$, $r^2 = 100$, *(a)* $a_1(t)$, *(b)* $a_2(t)$, *(c)* $u(t)$, *(d)* $x_1(t)$, *(e)* $x_2(t)$

With

$P_m = 1, P_e = 0.8, H = 0.45, K = -12, \alpha = -1.7, C = 2.4, \gamma_1 = -17, \gamma_2 = -13, \alpha = -2.7, r^2 = 81,$

the result was shown in Figure 37.

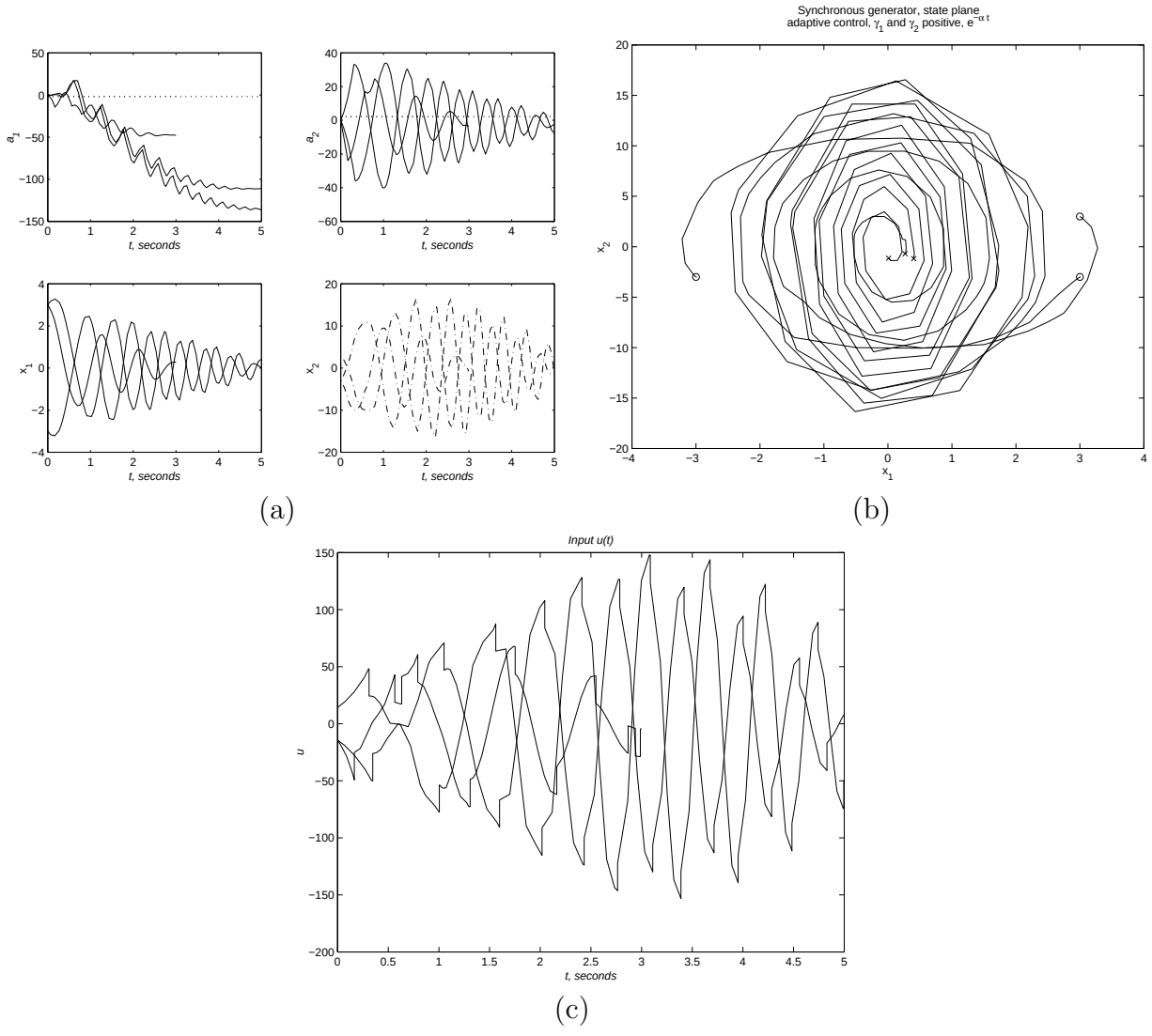


Figure 37: *Synchronous generator with adaptive control ($P_m = 1$, $P_e = 0.8$, $H = 0.45$, $K = -12$, $\alpha = -1.7$, $C = 2.4$, $\gamma_1 = -17$, $\gamma_2 = -13$, $\alpha = -2.7$, $r^2 = 81$, (a) $a_1(t)$, (b) $a_2(t)$, (c) $u(t)$, (d) $x_1(t)$, (e) $x_2(t)$*

Figure 37 has got initial conditions $(x_1, x_2) = (3, 3), (-3, -3), (3, -3)$ for (a) and (b), The file sizes were 21.68 KB for (a), 10.71 KB for (b), and 9.11 KB for (c).

Let us look again at the static part of the derivative of the Lyapunov function in Equation 80 and let us call it by the name of $\dot{V}_{stat.}$ By *static* we mean that there was neither $\bar{a}_1$ nor $\bar{a}_2$ terms.

$$\begin{aligned} \dot{V}_{stat.} &= C^2 x_1 x_2 + C\, x_2^2 + a_1 \left(C x_1 + x_2\right) \sin(x_1) + \\ &= a_2 P_m \left(C x_1 + x_2\right) + K\ \left(C\, x_1 + x_2\right) \mathrm{sgn}\left(\mathrm{C\,x_1 + x_2}\right). \end{aligned} \tag{92}$$

Figure 38 shows the case where Equation 92 is unstable ($\exists x_1, x_2$ s.t. $\dot{V}_{stat.} > 0$) and Figure 39 when it is stable ($\dot{V}_{stat.} < 0$).

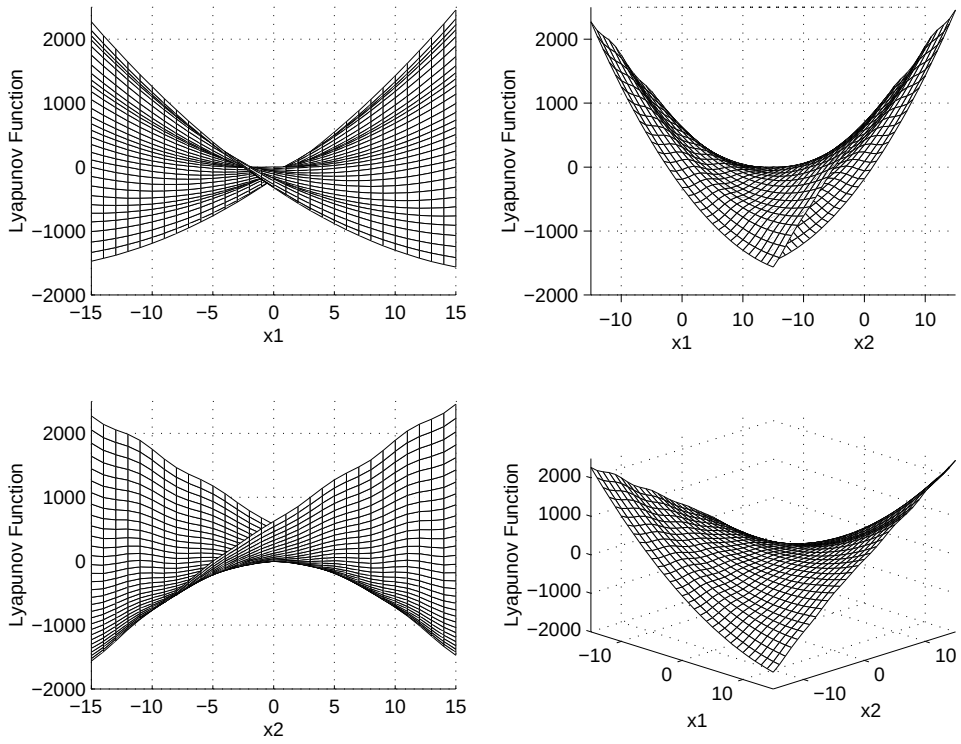


Figure 38: *The static part of the derivative of a Lyapunov function ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -5$, $C = 3$.) Here $\dot{V}_{stat.} = 2456.3$ at $(x_1, x_2) = (15, 15)$*

Figure 38 has $\dot{V}_{stat.} = 2456.3$ at $(x_1, x_2) = (15, 15)$, therefore is not stable.
Figure 39 shows that when the feed back gain K was changed from -5 as in Figure 38 to -3 the closed loop system became stabilizable since the sign of $\dot{V}$ then depended on the term that involved estimated parameters $\bar{a}_1$ and $\bar{a}_2$. In Figure 39 the $\max\left(\dot{V}_{stat.}\right) = 0$ occured at

$$(x_1, x_2) = \{(15, -5),\ (12, -4), (9, -3), (6, -2), (3, -1), (0, 0), (-3, 1), (-6, 2), (-9, 3), (-12, 4),\ (-15, 5)\}$$

ie on the hyperplane $s = 0$.

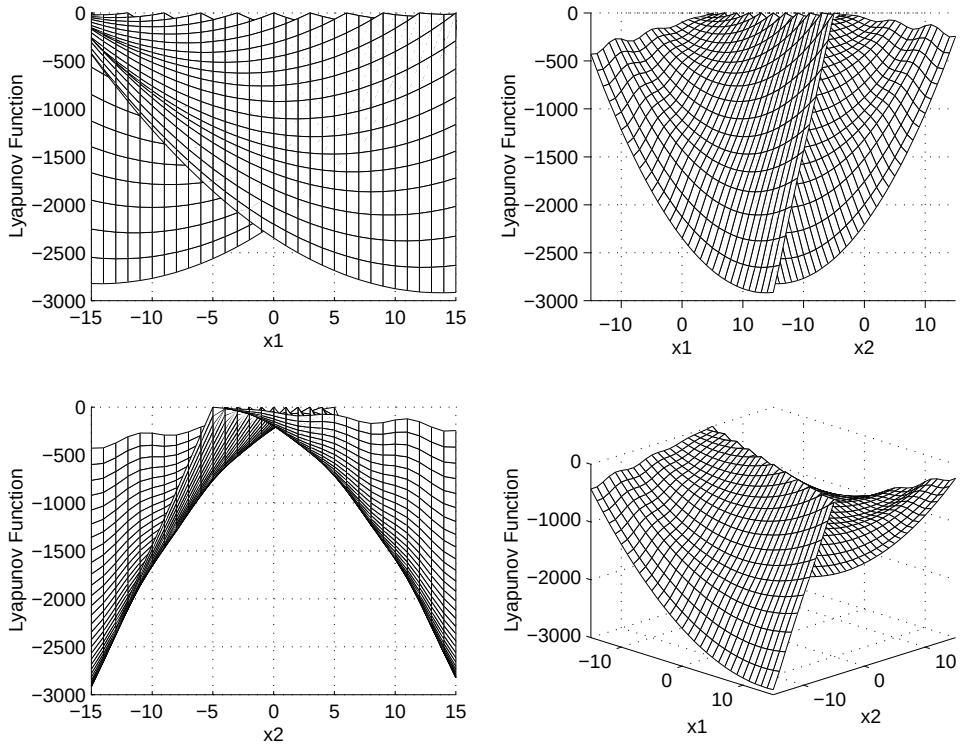


Figure 39: *The static part of the derivative of a Lyapunov function ($P_m = 2$, $P_e = 1.2$, $H = 1.3$, $K = -3$, $C = 3$.)*

Tracking control

For a system described by a state equation

$$\dot{x} = f(x) + g(u). \tag{93}$$

If the desired trajectory in phase plane be

$$\mathcal{F}(x_d) = 0, \tag{94}$$

where x_d is any point on the desired trajectory. It follows that

$$\mathcal{F}(x_d) = 0 \quad \Longleftrightarrow \quad x = x_d. \tag{95}$$

For the trajectory to remain on the desired path the following must be true,

$$V(x) > 0 \qquad \forall x \neq x_d \tag{96}$$
$$\dot{V}(x) < 0 \qquad \forall x \neq x_d \tag{97}$$

Choose the Lyapunov function as

$$V(x) = \mathcal{F}^2(x), \tag{98}$$

which has the derivative [2]

$$\dot{V}(x) = 2\mathcal{F}\mathcal{F}_x^T \dot{x} \tag{99}$$
$$= 2\mathcal{F}\mathcal{F}_x^T \left(f(x) + g(u)\right). \tag{100}$$

Let

$$g(u) = \mathcal{V}u. \tag{101}$$

The input is then

$$u = -K\,\mathrm{sgn}\left(\mathcal{F}\mathcal{F}_x^T\mathcal{V}\right), \tag{102}$$

where $\mathcal{V}$ is a vector satisfying $\mathcal{V}u$.
In our simplified model for synchronous generator

$$\left.\begin{array}{ll} \dot{x_1} & = x_2 \\ \dot{x_2} & = \frac{1}{H}Pm - \frac{1}{H}P_e \sin x_1 \end{array}\right\}, \tag{103}$$

let the desired trajectory be simply a circle centered at the origin

$$\mathcal{F}(x) = x_1^2 + x_2^2 - r^2, \quad r > 0. \tag{104}$$

[2] $f_x^T = \frac{df}{dx} = \left[\begin{array}{cccc} \frac{df}{dx_1} & \frac{df}{dx_2} & \cdots & \frac{df}{dx_n} \end{array}\right]^T$

From this $\mathcal{F}$ derive V and $\dot{V}$

$$\begin{aligned} V &= \mathcal{F}^2 \\ &= \left(x_1^2(t) + x_2^2(t) - r^2\right)^2, \qquad (105) \\ \dot{V} &= 2\mathcal{F}\mathcal{F}_x^T \dot{x} \\ &= 2\left(x_1^2(t) + x_2^2(t) - r^2\right)\left(2x_1(t)\dot{x}_1(t) + 2x_2(t)\dot{x}_2(t)\right) \qquad (106) \\ &= 4\left(\frac{1}{H}Pm - \frac{1}{H}P_e \sin x_1(t) + x_1(t)\right) x_2(t)\left(x_1^2(t) + x_2^2(t) - r^2\right). \qquad (107) \end{aligned}$$

Let the control input be

$$\begin{aligned} u &= -K\mathrm{sgn}\left(g\mathcal{F}\mathcal{F}_x^T\mathcal{V}_1\right) \\ &= -K\mathrm{sgn}\left(\frac{1}{H}\left(x_1^2(t) + x_2^2(t) - r^2\right)\begin{bmatrix} 2x_1(t) & 2x_2(t) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\ &= -K\mathrm{sgn}\left(\frac{2}{H}\left(x_1(t) + x_2(t)\right)\left(x_1^2(t) + x_2^2(t) - r^2\right)\right), \qquad (108) \end{aligned}$$

where $\mathcal{V}_1$ is a unit vector and here $g = \frac{1}{H}$.
Figure 40 (a) and (b) show the phase plane when the input was

$$u = -K\mathrm{sgn}\left(\frac{2}{H}\left(x_1(t) + x_2(t)\right)\left(x_1^2(t) + x_2^2(t) - r^2\right)\right) + \vartheta(\pm 1), \qquad (109)$$

where $\vartheta(\pm\varpi)$ was a uniform random number with value between $\pm\varpi$,

$$K = 3, \quad H = 0.8, \quad P_e = 1.3, \quad r = 2,$$

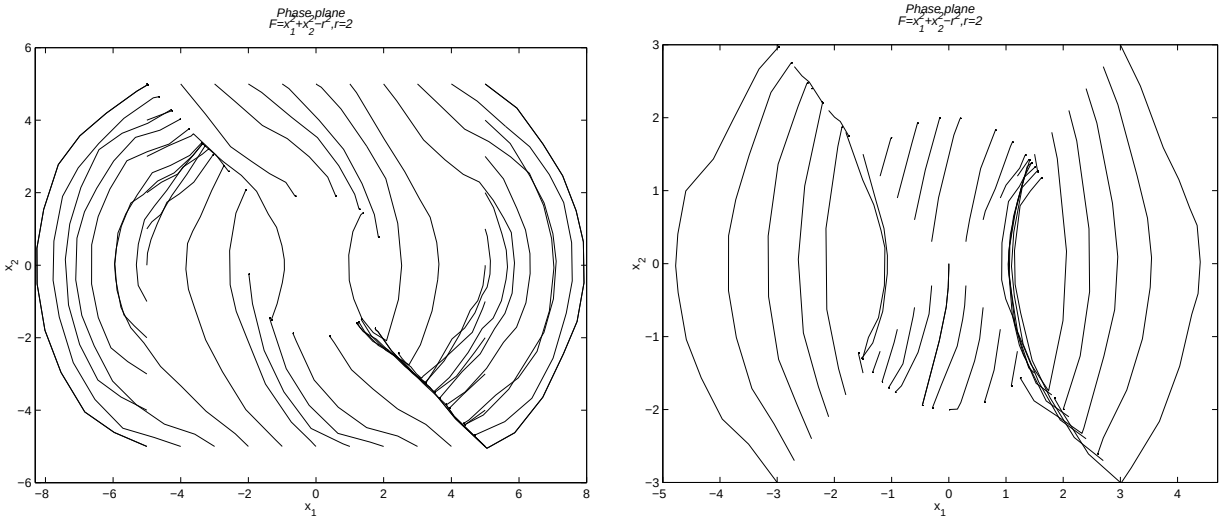
the initial conditions were for (a)

$$(x_1, x_2) = \{-5 \le x_1 \le 5\} \times \{-5, 5\} \bigcup \{-5, 5\} \times \{-5 \le x_2 \le 5\}$$

and for (b)

$$(x_1, x_2) = (\iota, \pm\iota), \iota = \{-3, -2.7, -2.4, \ldots, 3\}.$$

Miscellaneous details were, for (a) file size = 476.10 KB, for (b) $t_{CPU} = 1.57$ sec, file size = 519.76 KB.



(a) (b)

Figure 40: *Phase plane of a synchronous generator (2^{nd}-order) with a desired trajectory* $\mathcal{F}(x) = x_1^2 + x_2^2 - r^2$. *(a) time domain, (b) state space.*

Figure 41 shows the result when the input was

$$u = -K\mathrm{sgn}\left(\frac{2}{H}\left(x_1(t) + x_2(t)\right)\left(x_1^2(t) + x_2^2(t) - r^2\right)\right). \tag{110}$$

Initial conditons were, for (a)

$$(x_1, x_2) = \{-7 \leq x_1 \leq 7\} \times \{-7, 7\} \bigcup \{-7, 7\} \times \{-7 \leq x_2 \leq 7\},$$

and for (b)

$$(x_1, x_2) = (\iota, \pm\iota), \iota = \{-1.8, -1.6, -1.4, \ldots, 1.8\},$$

and for (c)

$$(x_1, x_2) = (7, 6).$$

Simulated with

$$K = 3, \quad H = 0.8, \quad P_e = 1.3, \quad r = 2.$$

The size of the files for Figure 41 (a) is 326.90 KB, for (b) is 286.50 KB, and for (c) is 41.84 KB.

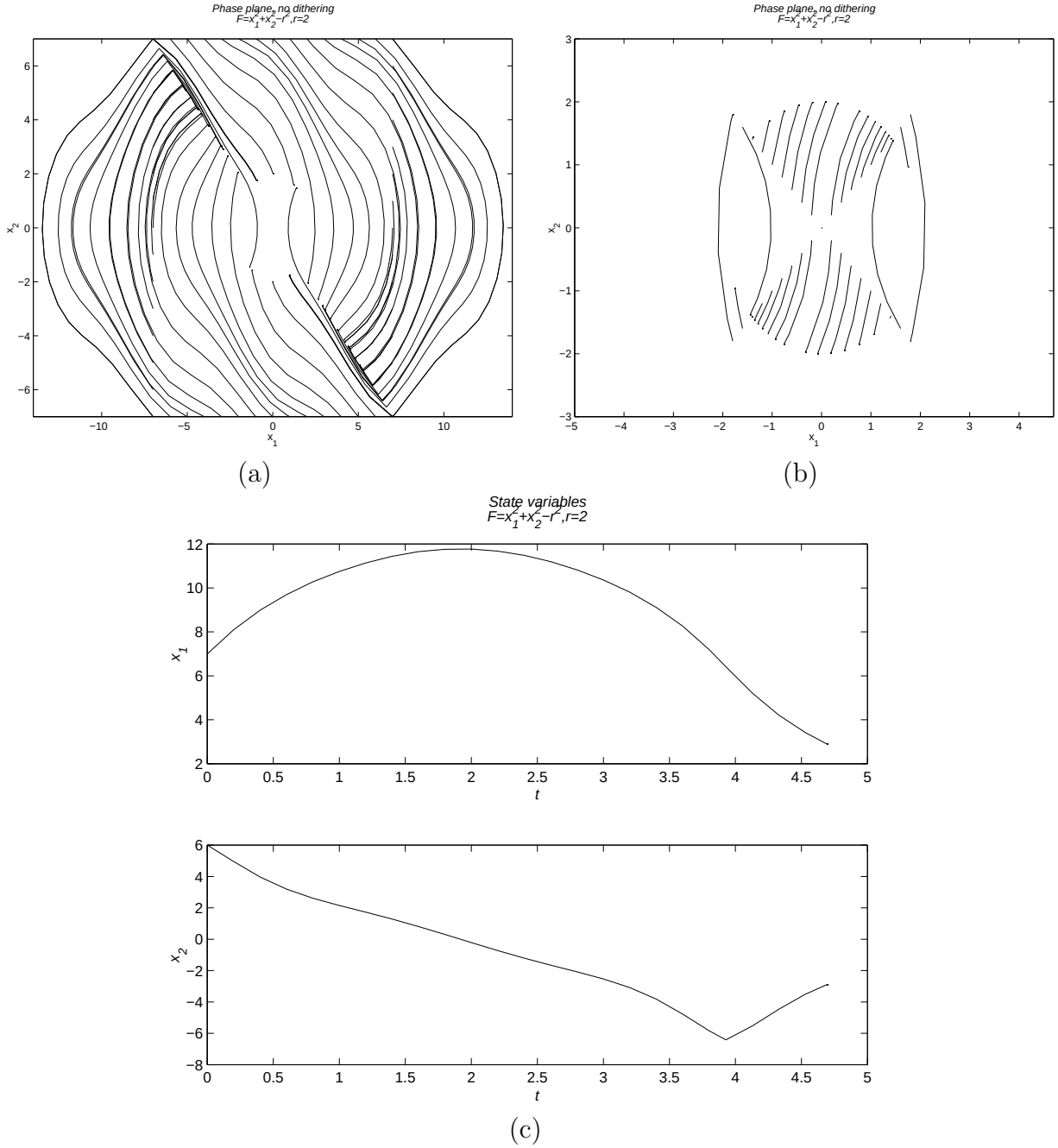


Figure 41: *Phase plane of a synchronous generator (2^{nd}-order) with a desired trajectory* $\mathcal{F}(x) = x_1^2 + x_2^2 - r^2$.

Let the control input be

$$\begin{aligned} u &= -K\mathrm{sgn}\left(\mathcal{F}\mathcal{F}_x^T\dot{x}\right) \\ &= -K\mathrm{sgn}\left(\left(x_1^2+x_2^2-r^2\right)\begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix}\begin{bmatrix} x_2 \\ \frac{1}{H}P_m - \frac{1}{H}P_e\sin x_1 \end{bmatrix}\right) \\ &= -K\mathrm{sgn}\left(\left(\frac{2x_2}{H}P_m - \frac{2x_2}{H}P_e\sin\left(x_1\right)+2x_1x_2\right)\left(x_1^2+x_2^2-r^2\right)\right). \end{aligned} \tag{111}$$

(112)

In the point of view of control, only the term having P_m can be considered, therefore

$$\begin{aligned} u &= -K\mathrm{sgn}\left(\left(x_1^2+x_2^2-r^2\right)\left(2x_2\frac{1}{H}\right)\right) \\ &= -K\mathrm{sgn}\left(\frac{2}{H}\left(x_2\right)\left(x_1^2(t)+x_2^2(t)-r^2\right)\right), \end{aligned} \tag{113}$$

Figure 42 shows the result from the simulation. The parameters were

$$K=-5.1, \quad H=0.7, \quad P_e=1, \quad r=2.3.$$

The initial conditions for Figure 42 were

top-left and -right $(x_1,x_2)=\{-4,4\}\times\{-4\le\mathcal{I}\le 4\}\bigcup\{-4\le\mathcal{I}\le 4\}\times\{-4,4\}$

bottom-left and -right $(x_1,x_2)=(i,i)\bigcup(i,-i), \quad i=-3\le\mathcal{I}\le 3$

Simulation had $t_{CPU}=22.93$ sec. file size = 938.29 KB.

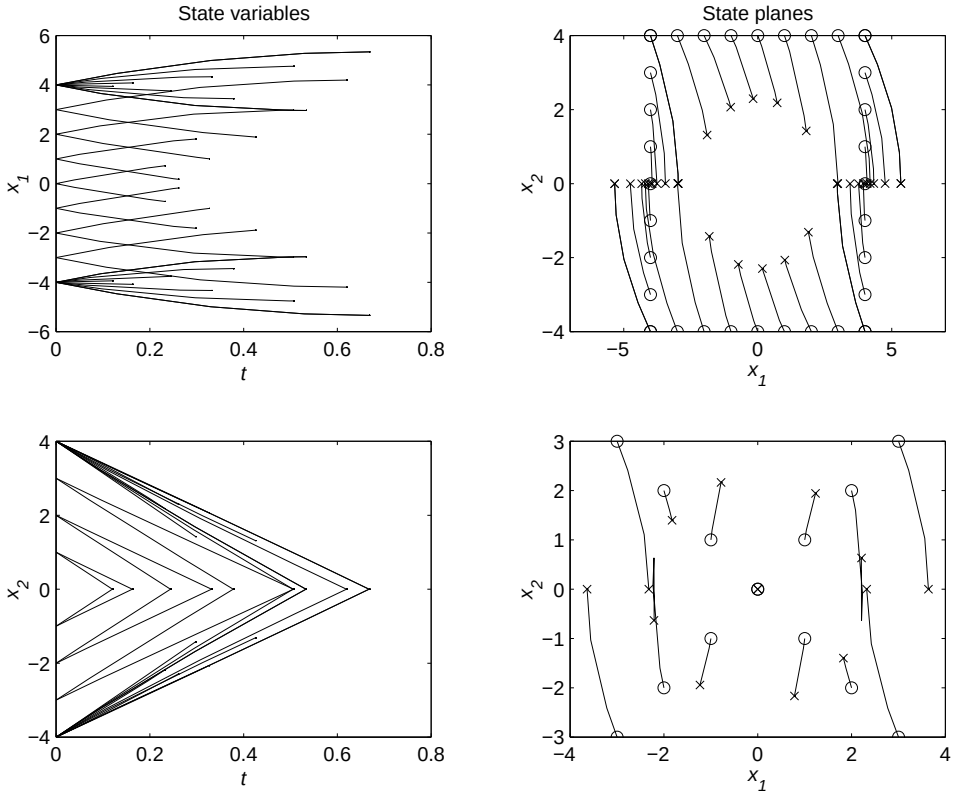


Figure 42: *Synchronous generator (2^{nd}-order) with a desired trajectory* $\mathcal{F}(x) = x_1^2 + x_2^2 - r^2$.

Next let the desired trajectory be described by

$$\mathcal{F} = \frac{(x_1(t)-1)^2}{1.5} + (x_2(t)-2)^2 + (x_1(t)-1)(x_2(t)-2) - 2.5^2, \tag{114}$$

which has got the trajectories along the root of the function $\mathcal{F} = 0$ or

$$x_2 = \frac{5 - x_1(t) \pm 1.29099\sqrt{14 + 2x_1(t) - x_1^2}}{2}. \tag{115}$$

The trajectory desired is shown in Figure 43. The Lyapunov function can be found as

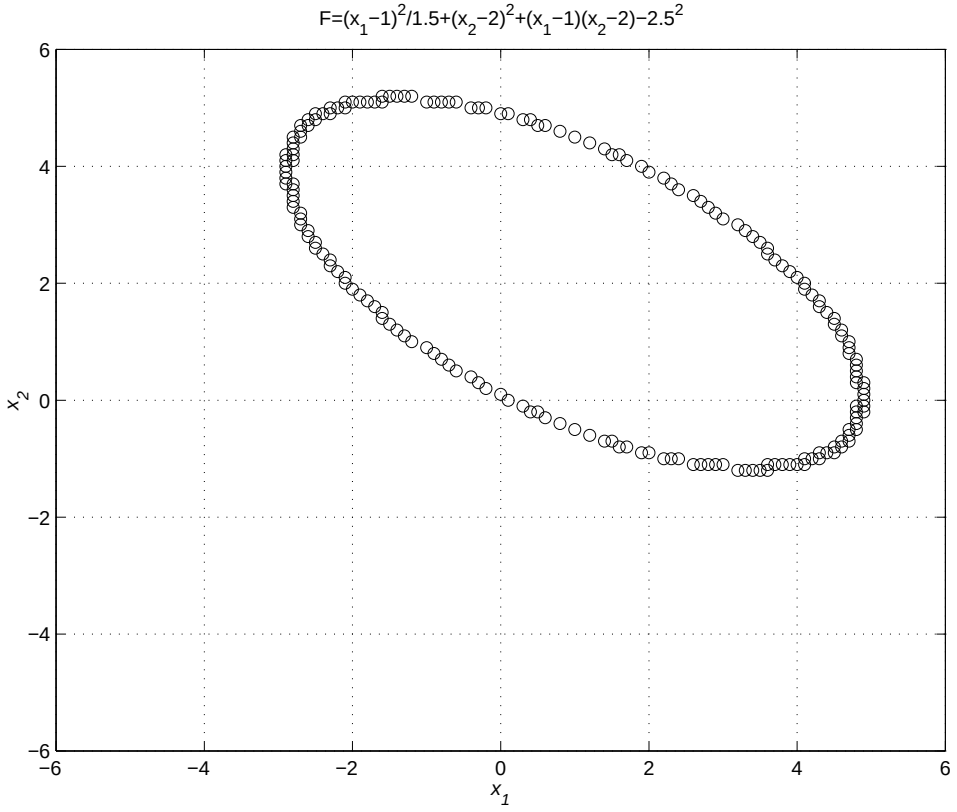


Figure 43: *The desired trajectory* $\mathcal{F} = \frac{(x_1(t)-1)^2}{1.5} + (x_2(t)-2)^2 + (x_1(t)-1)(x_2(t)-2) - 2.5^2$

$$V = \mathcal{F}^2 = \left(\frac{(x_1(t)-1)^2}{1.5} + (x_2(t)-2)^2 + (x_1(t)-1)(x_2(t)-2) - 2.5^2 \right)^2. \tag{116}$$

The derivative of $\mathcal{F}$ is

$$\begin{aligned}
\dot{\mathcal{F}} &= 2\mathcal{F}\mathcal{F}_x^T\dot{x} \\
&= 2\left(-6.25+\frac{1}{1.5}\left(-1+x_1^2\right)+(-1+x_1)(-2+x_2)+\left(-2+x_2^2\right)\right) \\
&\quad \left(\frac{4}{3}(-1+x_1)\dot{x}_1+(-2+x_2)\dot{x}_1+(-1+x_1)\dot{x}_2+2(-2+x_2)\dot{x}_2\right) && (117)\\
&= 2\left(-6.25+\frac{1}{1.5}(-1+x_1)^2+(-1+x_1)(-2+x_2)+(-2+x_2)^2\right)\left(\frac{4}{3}(-1+x_1)x_2+\right. \\
&\quad \left.(-2+x_2)x_2+(-1+x_1)(a_1-a_2x_1)+2(-2+x_2)(a_1-a_2x_1)\right) && (118)\\
&= 2\left(\frac{1}{2.4}+\frac{1}{1.5}x_1^2+x_1\left(-\frac{10}{3}+x_2\right)-5x_2+x_2^2\right) \\
&\quad \left(-5a_1-a_2x_1^2+\left(-\frac{10}{3}+2a_1\right)x_2+x_2^2+x_1\left(a_1+5a_2+\left(\frac{4}{3}-2a_2\right)x_2\right)\right) && (119)
\end{aligned}$$

Let the control input

$$\begin{aligned}
u &= -K\mathrm{sgn}\left(g\mathcal{F}\mathcal{F}_x^T\mathcal{V}_1\right) \\
&= -K\mathrm{sgn}\left(g\left(\frac{(x_1(t)-1)^2}{1.5}+(x_2(t)-2)^2+(x_1(t)-1)(x_2(t)-2)-2.5^2\right)\right. \\
&\quad \cdot 2\left(\frac{1}{2.4}+\frac{1}{1.5}x_1^2+x_1\left(-\frac{10}{3}+x_2\right)-5x_2+x_2^2\right) \\
&\quad \left.\left(-5a_1-a_2x_1^2+\left(-\frac{10}{3}+2a_1\right)x_2+x_2^2+x_1\left(a_1+5a_2+\left(\frac{4}{3}-2a_2\right)x_2\right)\right)\right) && (120)\\
&= -K\mathrm{sgn}\left(\frac{2}{H}\left(-6.25+\frac{1}{1.5}\left(-1+x_1^2\right)+(-1+x_1)(-2+x_2)+(-2+x_2)^2\right)\left(\frac{1}{2.4}+\right.\right. \\
&\quad \frac{1}{1.5}x_1^2+x_1\left(-\frac{10}{3}+x_2\right)-5x_2+x_2^2\Big)\left(-5a_1-a_2x_1^2+\left(-\frac{10}{3}+2a_1\right)x_2+\right. && (121)\\
&\quad \left.\left.x_2^2+x_1\left(a_1+5a_2+\left(\frac{4}{3}-2a_2\right)x_2\right)\right)\right), && (122)
\end{aligned}$$

which, when

$$P_m=2,\quad P_e=1.36,\quad g=\frac{1}{H}=\frac{1}{0.8}=1.25,\quad \Longrightarrow a_1=2.5,\quad a_2=1.7,$$

turns out to be

$$\begin{aligned}
u &= -K\mathrm{sgn}\left(\frac{1}{H}\left(2\left(-6.25+\frac{1}{1.5}(-1+x_1)^2+(-1+x_1)(-2+x_2)+(-2+x_2)^2\right)\right.\right. \\
&\quad \left(\frac{1}{2.4}+\frac{1}{1.5}x_1^2+x_1\left(-\frac{10}{3}+x_2\right)-5x_2+x_2^2\right) \\
&\quad \left.\left.\left(-12.5-1.7x_1^2+x_1(11-2.06667x_2)+\frac{5}{3}x_2+x_2^2\right)\right)\right) && (123)
\end{aligned}$$

$$= -K\text{sgn}\left(2.5\left(\frac{1}{2.4}+\frac{1}{1.5}x_1^2+x_1\left(-\frac{10}{3}+x_2\right)-5x_2+x_2^2\right)\right.$$
$$\left.\left(-12.5-1.7x_1^2+x_1\left(11.-2.06667x_2\right)+\frac{5}{3}x_2+x_2^2\right)\right) \quad (124)$$

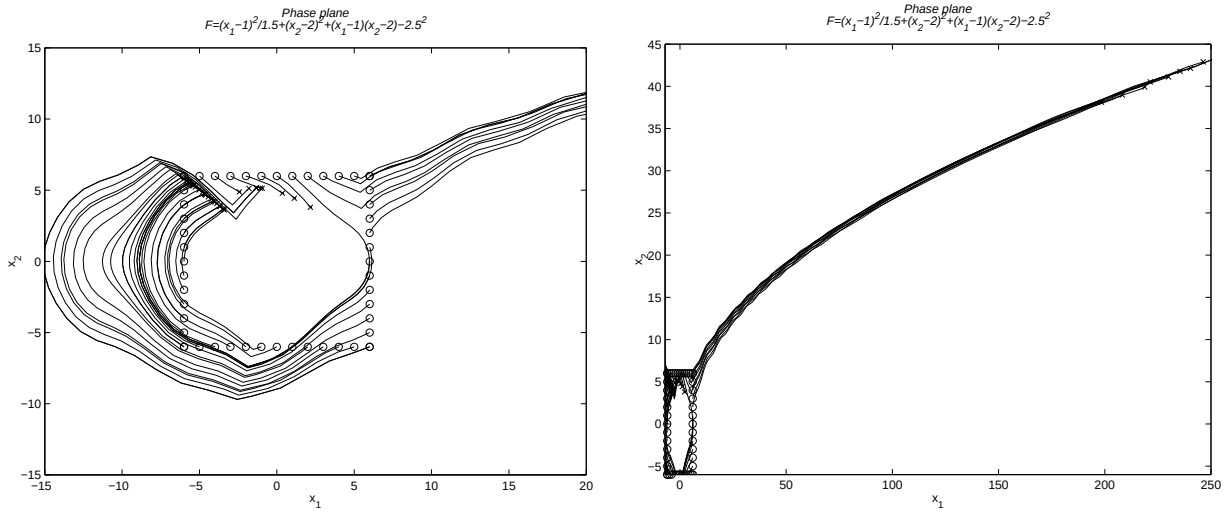
Figure 44 shows the result from a simulation where the desired trajectory was

$$\mathcal{F}=\frac{\left(x_1-1\right)^2}{1.5}+\left(x_2-2\right)^2+\left(x_1-1\right)\left(x_2-2\right)-2.5^2,$$

and the initial conditions were

$$\left(x_1,x_2\right)=\left\{-6\leq\mathcal{I}\leq 6\right\}\times\left\{-6\leq\mathcal{I}\leq 6\right\}.$$

Files of Figure 44 (a) was 232.53 KB, and (b) was 234.50 KB.



(a) (b)

Figure 44: *State plane of a synchronous generator (2^{nd}-order) with a desired trajectory* $\mathcal{F}=\frac{\left(x_1-1\right)^2}{1.5}+\left(x_2-2\right)^2+\left(x_1-1\right)\left(x_2-2\right)-2.5^2$

Let

$$\mathcal{F} = 4x_1^2 + x_2^2 - r^2 = 0, \tag{125}$$

which can be drawn as Figure 45.

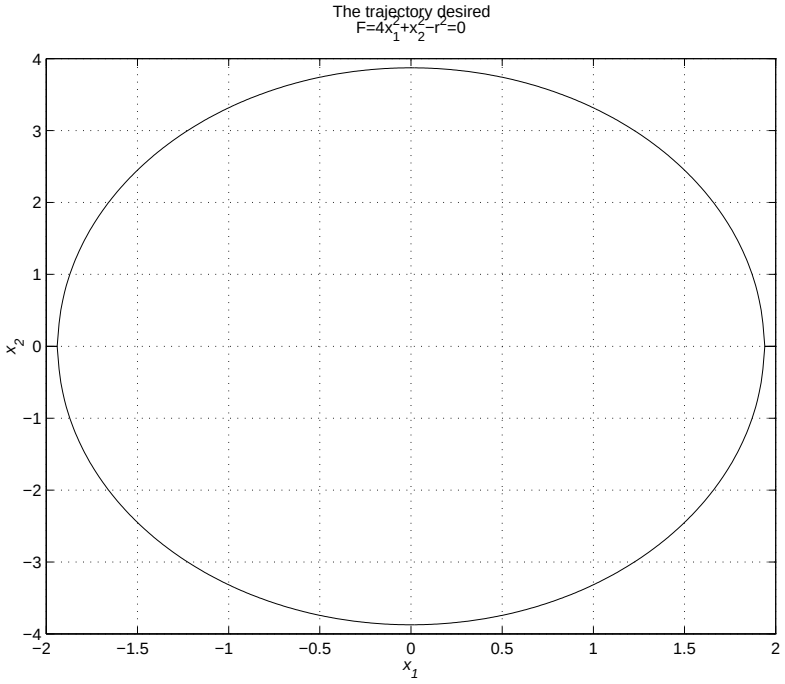


Figure 45: *The designed trajectory* $\mathcal{F} = 4x_1^2 + x_2^2 - r^2 = 0$.

$$\begin{aligned} u &= -K\mathrm{sgn}\left(\mathcal{F}\mathcal{F}_x^T\dot{x}\right) \\ &= -K\mathrm{sgn}\left(\left(4x_1^2 + x_2^2 - r^2\right)\begin{bmatrix} 8x_1 & 2x_2 \end{bmatrix}\begin{bmatrix} x_2 \\ \frac{1}{H}P_m - \frac{1}{H}P_e\sin x_1 \end{bmatrix}\right) \\ &= -K\mathrm{sgn}\left(\left(\frac{2x_2}{H}P_m - \frac{2x_2}{H}P_e\sin(x_1) + 8x_1x_2\right)\left(4x_1^2 + x_2^2 - r^2\right)\right). \end{aligned} \tag{126}$$

$$\tag{127}$$

Only the term containing P_m is useful for controlling purpose, thus let

$$u = -K\mathrm{sgn}\left(\left(\frac{2x_2}{H}\right)\left(4x_1^2 + x_2^2 - r^2\right)\right). \tag{128}$$

Then with the system having

$$K = -10.5, \quad H = 0.5, \quad P_e = 1, \quad r = 15,$$

and the initial conditions of

$$(x_1, x_2) = (\varpi, \varpi), (\varpi, -\varpi), \quad \varpi = \{-20, -13, -6, 1, 8, 15\}$$

did the simulation and obtain the result of Figure 46. The size of the files for Figure 46 (a) is 249.15 KB, (b) 270.51 KB, (c) 416.25 KB, and (d) 271.37 KB.

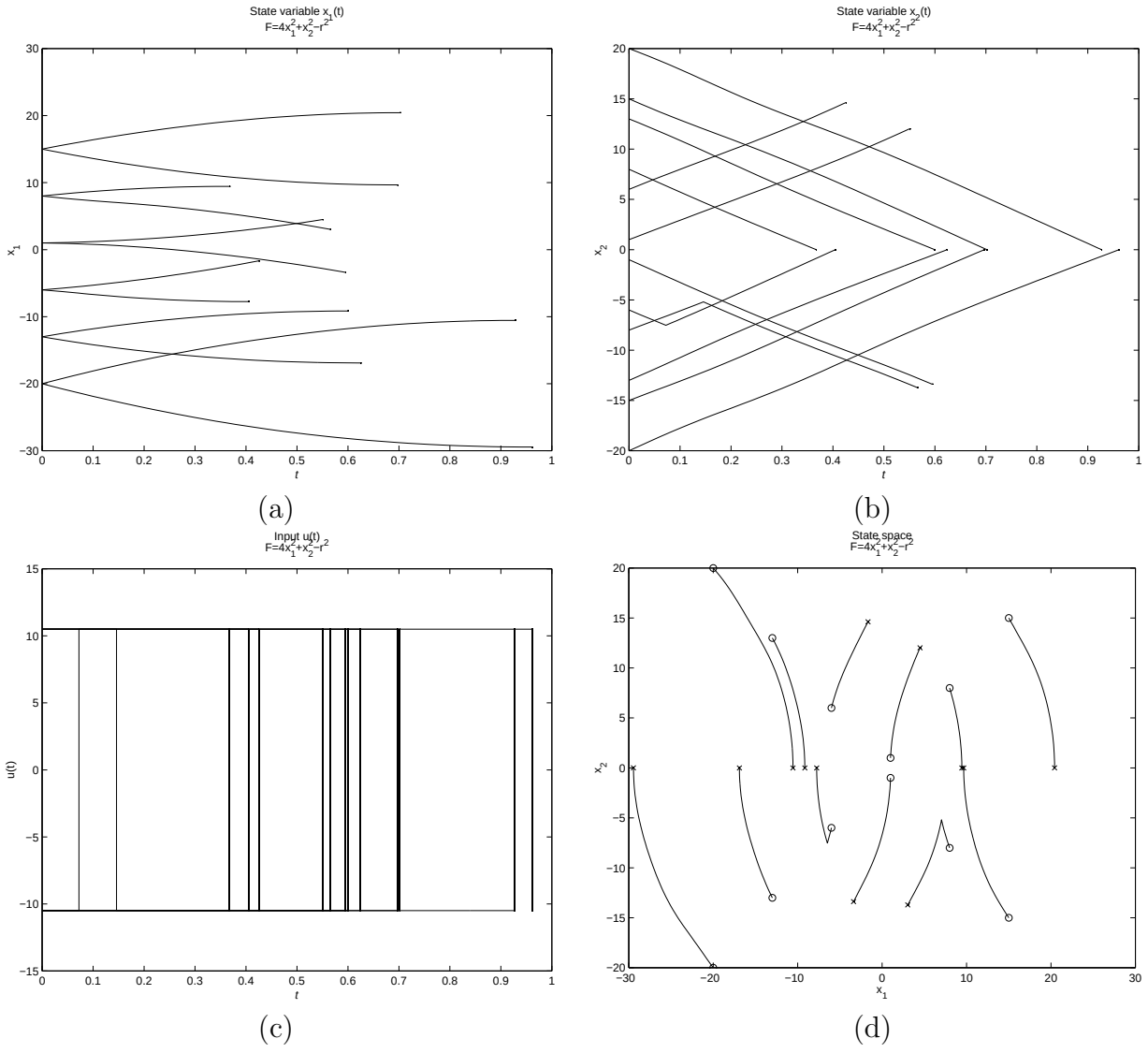


Figure 46: *State plane of a synchronous generator (2^{nd}-order) with a desired trajectory* $\mathcal{F} = 4x_1^2 + x_2^2 - r^2 = 0$. *(a)* $x_1(t)$, *(b)* $x_2(t)$, *(c)* $u(t)$, *(d)* x_2 *v.s.* x_1.

Figure 46 (c) shows how the input has got a bang-bang characteristic (with a subtle switching logic). In Figure 46 (d) one can see that the trajectories either converge to the $x_2 = 0$ line or converge onto the designed trajectory. The problem still remain is how to make the trajectory move along once the designed trajectory is reached. (because at the moment the trajectories only go to the desired curve and then rest there)

Discussion

This section is for discussions in addition to those already written above with the result of simulation.
From Figure 14 the equilibrium point is approximately $(1 \pm 2n\pi, 0)$ where $n = 0, 1, 2, \ldots$. Both Equation 50 and Equation 51 gives eigenvalues in the same form

$$\mp a \pm \jmath 1$$

and both gives a similar trajectory, therefore we may be able to add this form of eigenvalues to the standard stereotypes which are used as a tool for classifying trajectories. (These stereotypes are namely

1. $\lambda_1 \neq \lambda_2 \neq 0, \quad \lambda_1, \lambda_2 \in \mathcal{R}$: which gives the equilibrium point as
   (a) *a stable node* if $\lambda_2 < \lambda_1 < 0$
   (b) *an unstable node* if $\lambda_2 > \lambda_1 > 0$, and
   (c) *a saddle point* if $\lambda_2 < 0 < \lambda_1$
2. $\lambda_{1,2} = \alpha \pm \jmath\beta, \quad \alpha, \beta \in \mathcal{R}$: which gives the equilibrium point as
   (a) *a stable focus* if $\alpha < 0$
   (b) *an unstable focus* if $\alpha > 0$, and
   (c) *a center* if $\alpha = 0$
3. $\lambda_1 = \lambda_2 = \lambda \neq 0$: which which gives the equilibrium point as
   (a) *a stable node* if $\lambda < 0$, and
   (b) *an unstable node* if $\lambda > 0$
4. $\lambda_1 = 0$: a degenerate case where all trajectories
   (a) converge to an equlibrium subspace if $\lambda_2 < 0$, and
   (b) diverge to an equlibrium subspace if $\lambda_2 > 0$
5. $\lambda_1 = \lambda_2 = 0$: a degenerate case where every point in the plane is an equilibrium point.)

To find a region of attraction one linearizes a system

$$\dot{x} = f(x) \tag{129}$$

about an equilibrium point x_{eq}

$$\dot{x} = Ax, \quad A = \frac{\partial f}{\partial x}(x_{eq}). \tag{130}$$

Then if we let

$$\Lambda = \text{the set of all eigenvalues of } A,$$

then

$$\forall \lambda_i \in \Lambda \quad \operatorname{Re}\lambda_i < 0 \implies x_{eq} \text{ is asymptotically stable} \tag{131}$$
$$\exists \lambda_i \in \Lambda \quad \operatorname{Re}\lambda_i > 0 \implies x_{eq} \text{ is unstable.} \tag{132}$$

A region of attraction is only meaningful for an asymptotically stable system. Both Equation 50 and 51 has got $\lambda_i \in \Lambda$ such that $\operatorname{Re}\lambda_i > 0$ therefore neither are stable, thus we may not find a domain of attraction for either system. This fact was confirmed by the five unsuccessful Lyapunov candidates described in Equation 54, 58, 61, 63, and 65. These Lyapunov candidates have the derivatives which are plotted in Figure 17, 18, 19, and 20 for Equation 54, 61, 63, and 65 respectively.
The stability of Equation 129 can be described as being one of these types ([Kha96], Definition 3.2). (Notice that *uniform* means independence of t_0 and *asymptotic* means $x_t \to 0$ as $t \to \infty$.)

Stable if,

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon, t_0) > 0 \text{ s.t. } ||x_{t_0} - x_{eq.}|| < \delta \implies ||x_t - x_{eq.}|| < \epsilon, \forall t \geq t_0 \geq 0. \tag{133}$$

Uniformly stable if,

$$\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0 \text{ s.t. } ||x_{t_0} - x_{eq.}|| < \delta \implies ||x_t - x_{eq.}|| < \epsilon, \forall t \geq t_0 \geq 0. \tag{134}$$

Unstable if not stable.

Asymptotically stable if stable and,

$$\exists c = c(t_0) > 0 \text{ s.t. } x_t \to 0 \text{ as } t \to \infty, \forall\, ||x_{t_0} - x_{eq.}|| < c. \tag{135}$$

Uniformly asymptotically stable if uniformly stable and,

$$\exists c > 0 \text{ s.t. } \forall \epsilon > 0, \exists T = T(\epsilon) > 0 \text{ s.t. } ||x(t) - x_{eq.}|| < \epsilon, \forall t \geq t_0 + T_\epsilon, \forall\, ||x_{t_0} - x_{eq.}|| < c. \tag{136}$$

Globally uniformly asymptotically stable if uniformly stable and,

$$\forall \epsilon, \forall c \in \Re^+, \exists T = T(\epsilon, c) > 0 \text{ s.t. } ||x(t) - x_{eq.}|| < \epsilon, \forall t \geq t_0 + T_{\epsilon,c}, \forall\, ||x_{t_0} - x_{eq.}|| < c. \tag{137}$$

The above description can be summarized as in Figure 47.

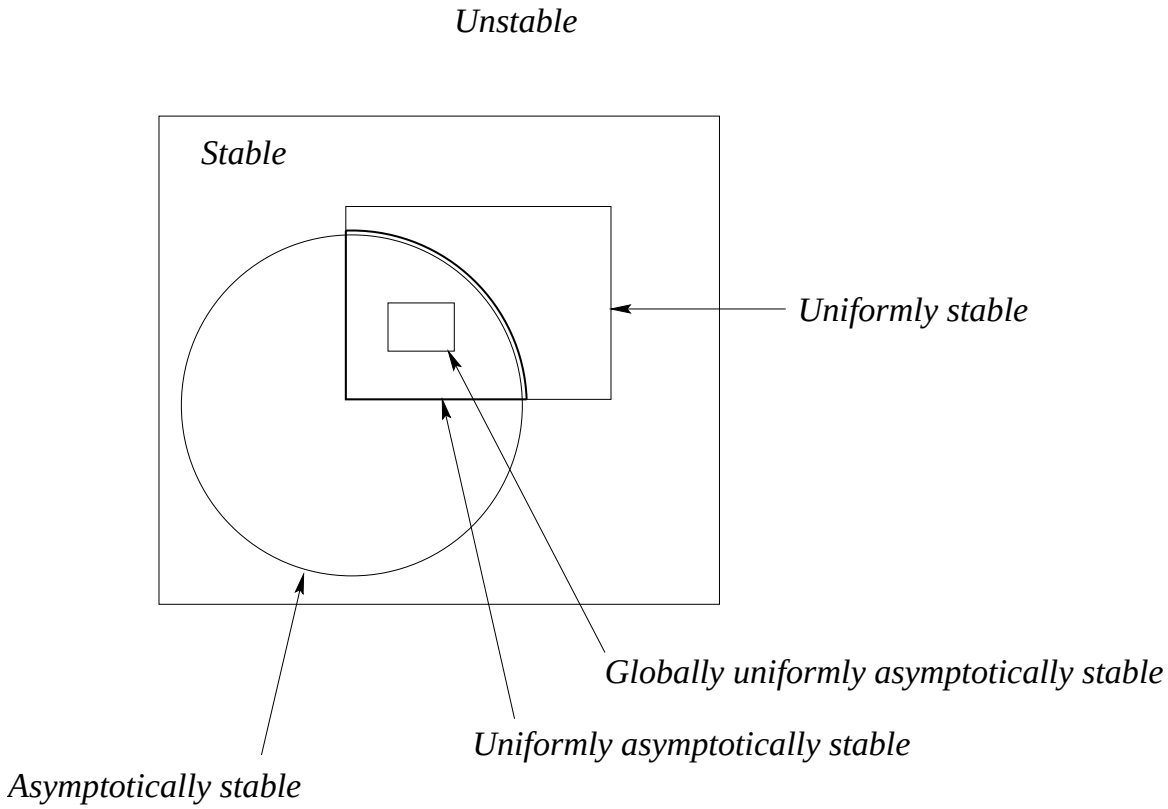


Figure 47: *Types of stability*

Synchronous generator, a model with three state variables

A more complicated model of a synchronous generator connected to an infinite bus can be represented by [Kha96]

$$H\ddot{\delta} = P_m - D\dot{\delta} - \eta_1 P_e \sin\delta \tag{138}$$
$$\tau\dot{P}_e = -\eta_2 P_e + \eta_3 \cos\delta + E_{FD}, \tag{139}$$

where H is inertial coefficient, δ an angle in radians, P_m mechanical input power, D damping coefficient, $\eta_1, \eta_2, \eta 3$ are constant parameters, P_e voltage, τ a time constant and E_{FD} field voltage (input.)

Relationship with the simplified model and a pendulum equation

How this model is related to the previously described simplified model can be explained as the following. If τ is relatively large, then

$$\dot{P}_e \approx 0, \quad \text{i.e.} \quad P_e = \text{constant}.$$

Then

$$H\ddot{\delta} = P_m - D\dot{\delta} - \eta_1 P_e \sin(\delta), \tag{140}$$

which is a pendulum equation. And if damping is also ignored then $D = 0$ or

$$H\ddot{\delta} = P_m - \eta_1 P_e \sin(\delta), \tag{141}$$

which is our simplified synchronous generator model.

Example of the response of the system

By letting

$$\delta = x_1, \quad \dot{\delta} = x_2, \quad P_e = x_3 \tag{142}$$

the state equation of this system is represented by

$$\dot{x_1} = x_2 \tag{143}$$
$$\dot{x_2} = \frac{P_m}{H} - \frac{D}{H}x_2 - \frac{\eta_1}{H}x_3 \sin\, x_1 \tag{144}$$
$$\dot{x_3} = -\frac{\eta_2}{\tau}x_3 + \frac{\eta_3}{\tau}\cos\, x_1 + \frac{E_{FD}}{\tau} \tag{145}$$

or

$$\dot{x_1} = x_2 \tag{146}$$
$$\dot{x_2} = a_1 + a_2 x_2 + a_3 x_3 \sin\, x_1 \tag{147}$$
$$\dot{x_3} = a_4 x_3 + a_5 \cos\, x_1 + a_6 \tag{148}$$

where

$$a_1 = \frac{P_m}{H}, a_2 = -\frac{D}{H}, a_3 = -\frac{\eta_1}{H}, a_4 = -\frac{\eta_2}{\tau}, a_5 = \frac{\eta_3}{\tau}, a_6 = \frac{E_{FD}}{\tau},$$

Let

$$H = 0.02, P_m = 0.9, \frac{D}{H} = 5, \eta_1 = 2.2, \eta_2 = 2.5, \eta 3 = 1.9, \tau = 7, E_{FD} = 1.3,$$

and obtain an explicit equation of states

$$\dot{x_1} = x_2 \tag{149}$$

$$\dot{x_2} = 45 - 5\,x_2 - 110\,x_3 \sin\, x_1 \tag{150}$$

$$\dot{x_3} = -0.3571\,x_3 + 0.2714\,\cos\, x_1 + 0.1857 \tag{151}$$

The result is plotted in Figure 48. Here the initial conditions were

$$(x_1, x_2, x_3) = (1, 1, 1), (-1, -1, -1), (3, -3, 3), (-3, 3, -3)$$

for Figure 48 (a) and (b), and

$$(x_1, x_2, x_3) = (1, -3, -2), (-1, 2, 6), (5, -3, 4), (2, 2, -3),$$

for (c) and (d). The CPU time for (a) and (b) was 0.13 seconds, and for (c) and (d) was 0.16 seconds.

Figure 48 is a synchronous generator since $P_m > 0$. From Figure 48 ((a) to (d)) one can see that

$x_3 = P_e > 0$ resulted in a stable system, ie. a generator supplying electric power is stable.

$x_3 = P_e < 0$ resulted in an unstable system, ie. when injecting electric power into a running generator it would become unstable.

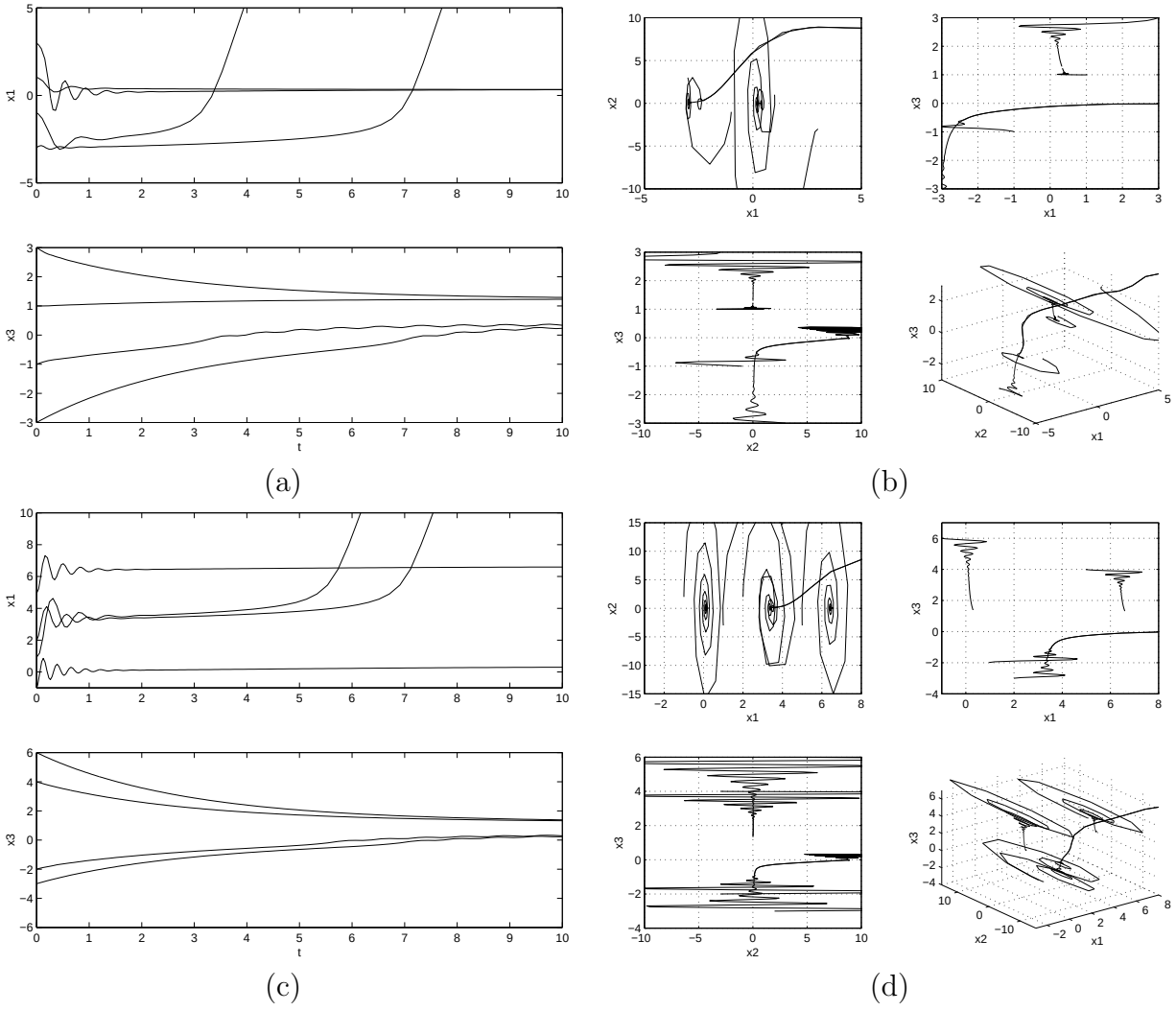


Figure 48: *Synchronous generator (a) state variables, (b) state plane, (c)* $x_1(t)$ *and* $x_3(t)$*, (d) state space.*

$\text{sgn}(\delta)$	$\text{sgn}\left(\dot{\delta}\right)$	$\text{sgn}(P_e)$	Figure #
$+$	$+$	$+$	Figure 49
$+$	$+$	$-$	Figure 50
$+$	$-$	$+$	Figure 51
$+$	$-$	$-$	Figure 52
$-$	$+$	$+$	Figure 53
$-$	$+$	$-$	Figure 54
$-$	$-$	$+$	Figure 55
$-$	$-$	$-$	Figure 56

Table 3: *Sign table for the investigation to study the effect of initial conditions on response*

Initial conditions #	$\lvert x_1\rvert$	$\lvert\dot{\delta}\rvert$	$\lvert P_e\rvert$
# 1	$\text{sgn}(\delta)\times 1$	$\text{sgn}\left(\dot{\delta}\right)\times 2$	$\text{sgn}(P_e)\times 3$
# 2	$\text{sgn}(\delta)\times 3$	$\text{sgn}\left(\dot{\delta}\right)\times 1$	$\text{sgn}(P_e)\times 2$
# 3	$\text{sgn}(\delta)\times 2$	$\text{sgn}\left(\dot{\delta}\right)\times 3$	$\text{sgn}(P_e)\times 1$
# 4	$\text{sgn}(\delta)\times 5$	$\text{sgn}\left(\dot{\delta}\right)\times 5$	$\text{sgn}(P_e)\times 5$
# 5	$\text{sgn}(\delta)\times 0.1$	$\text{sgn}\left(\dot{\delta}\right)\times 0.1$	$\text{sgn}(P_e)\times 0.1$

Table 4: *The initial conditons to study the effect of initial conditions on response*

Next is an investigation of the response of this system with all the possible combinations of state variable signs. Table 3 and Table 4 summarize the plan for simulation. The system simulated was the system described in Equation 143, 144, and 145 with

$$P_m = 1, \quad H = 0.05, \quad \eta_1 = 2.1, \quad \eta_2 = 2.7, \quad \eta_3 = 1.8, \quad \tau = 10, \quad E_{FD} = 1.32.$$

The CPU time for Figure 49 was 0.09 seconds, for Figure 50 was 0.15 seconds, for Figure 54 was 0.1 seconds, for Figure 55 was 0.11 seconds, and for Figure 56 was 0.13 seconds.
From the result we can see that with regard to neither $\text{sgn}(\delta)$ nor $\text{sgn}\left(\dot{\delta}\right)$, when $P_e(0) < 0$ the system would become unstable (ie. δ increase unboundedly with time). Vice versa, when $P_e > 0$ the system normally become stable afterward (unless when $|P_e|$ is small, for example $|P_e| = 0.1$)
The following are figures of the result.
Errata: the labels written on the figures (ie. Figure 49, 50, 51, 52, 53, 54, 55, and 56) was wrong, $P_m(0)$ should be replaced by $\delta(0)$, $P_e(0)$ by $\dot{delta}(0)$, and $\delta(0)$ by $P_e(0)$. However, the captions which are written next to the figures' number are correct.

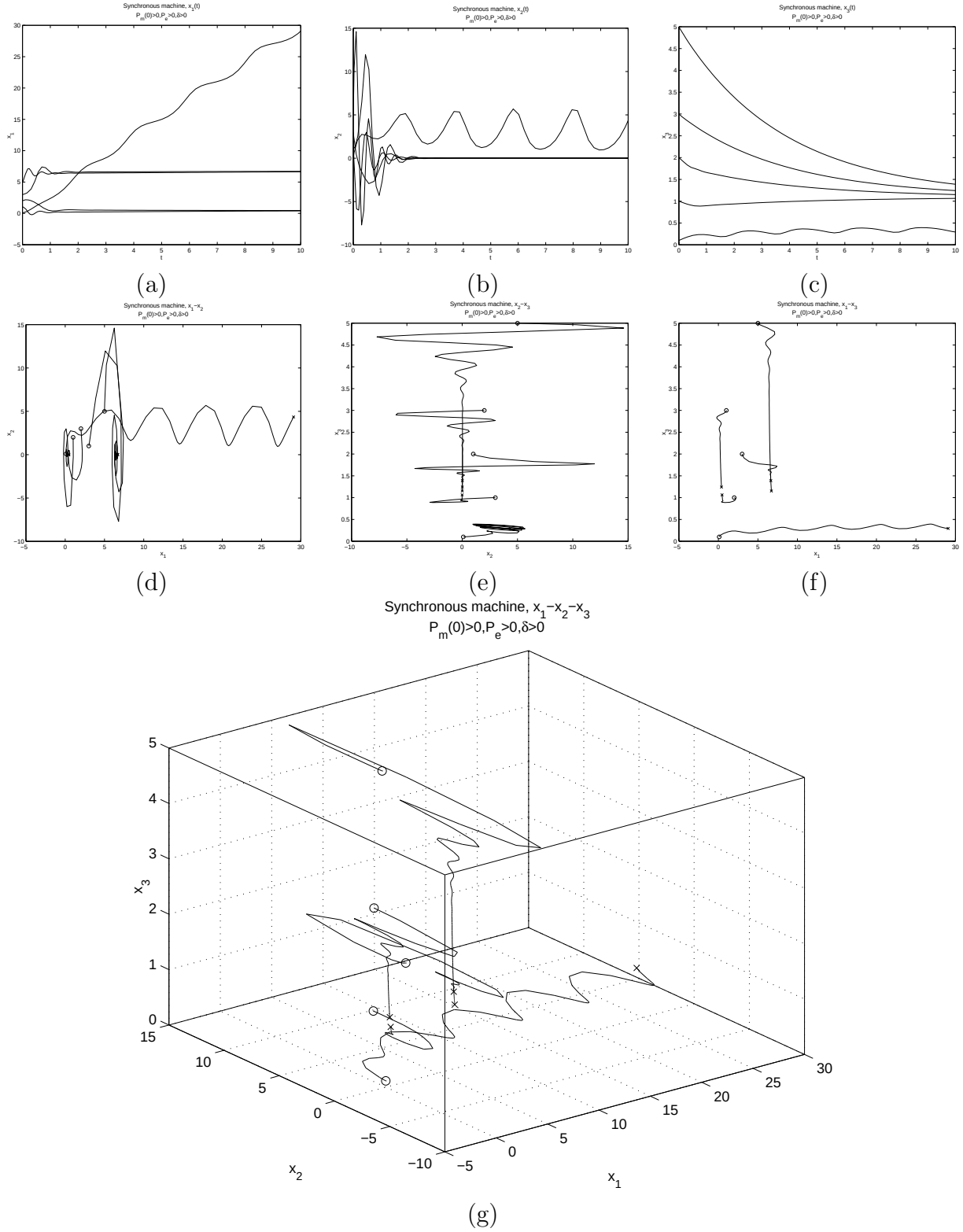
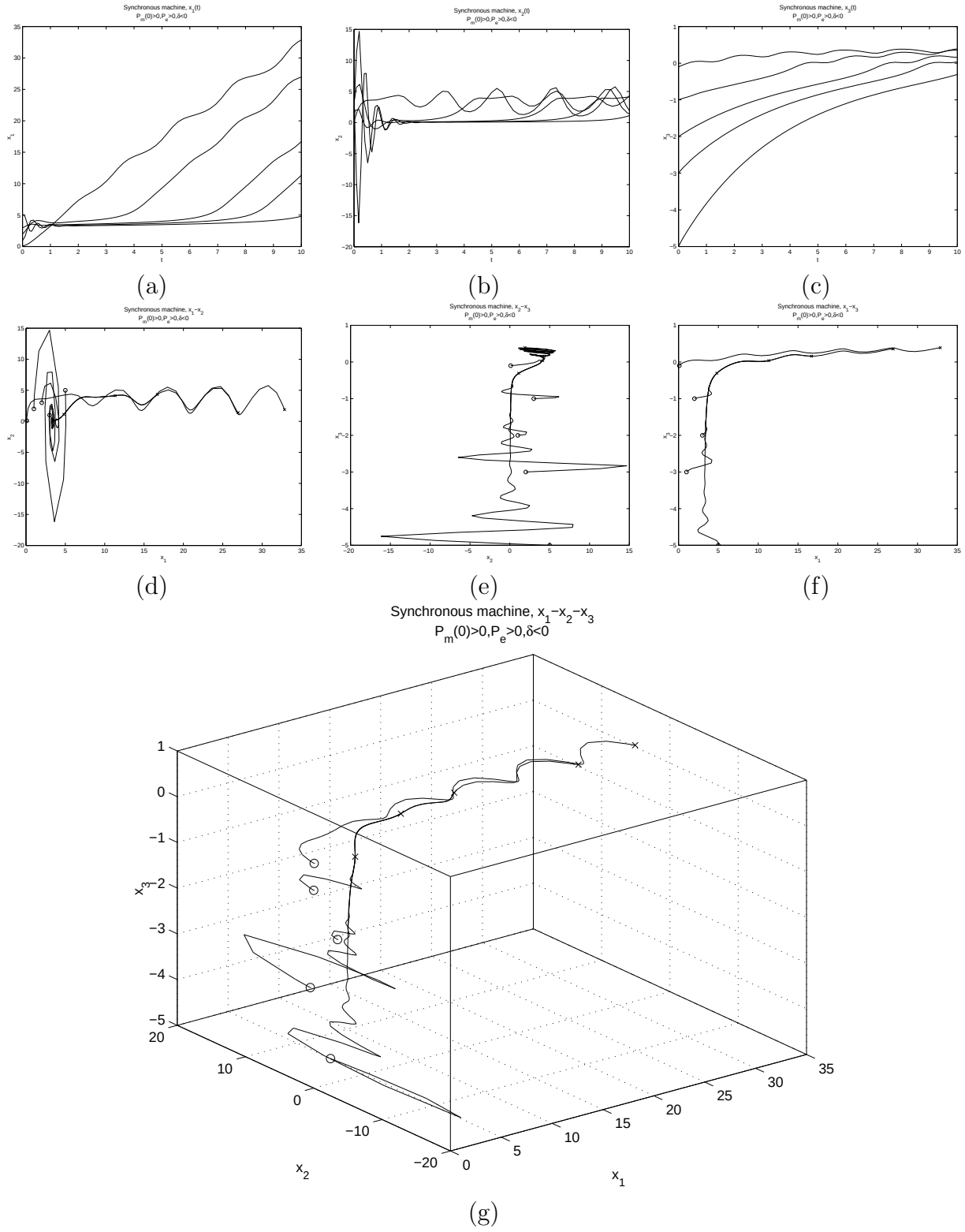


Figure 49: $\delta(0) > 0$, $\dot{delta}(0) > 0$, $P_e(0) > 0$. *(see Errata above)*



(a) (b) (c)

(d) (e) (f)

(g)

Figure 50: $\delta(0) > 0$, $\dot{delta}(0) > 0$, $P_e(0) > 0$.

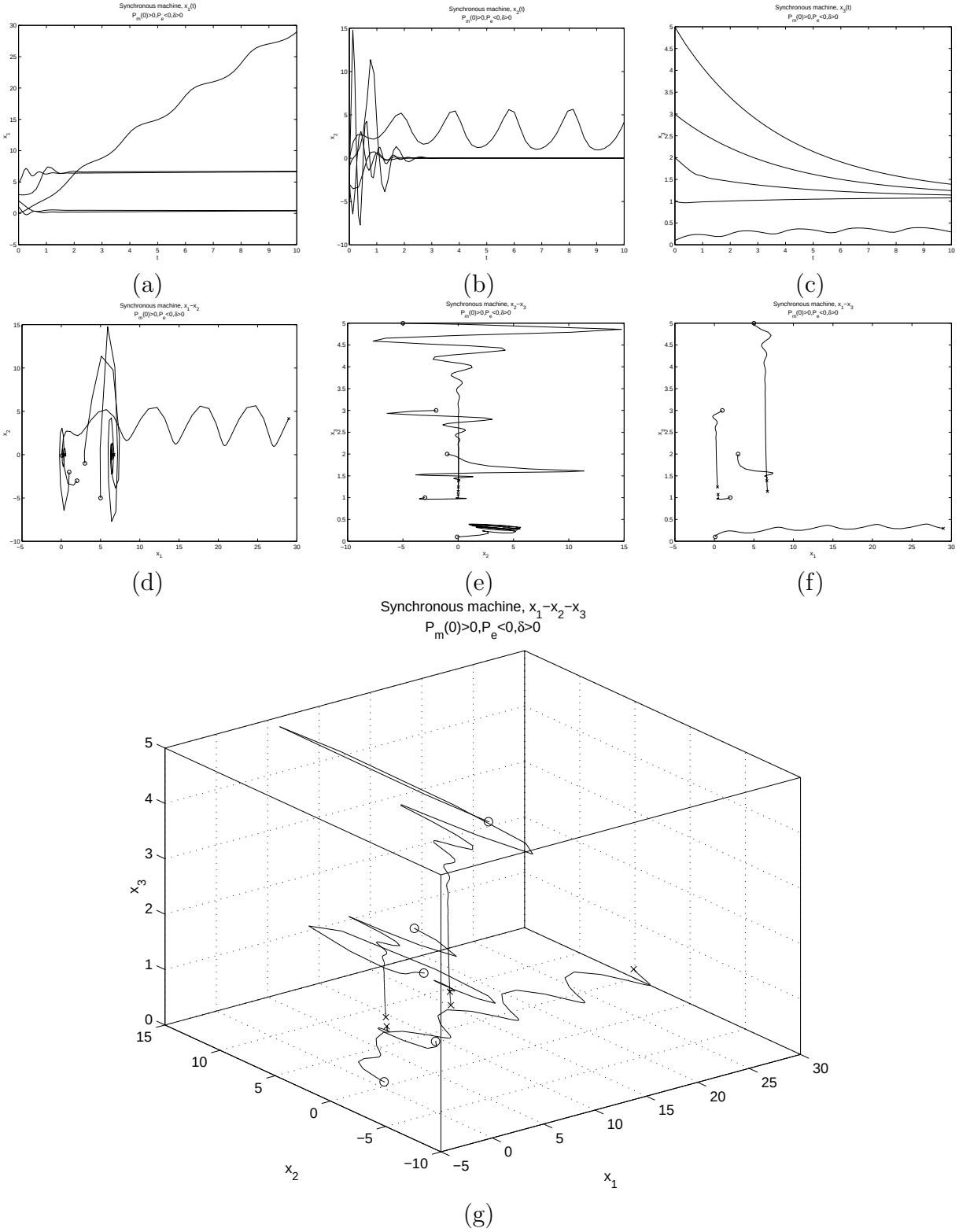


Figure 51: $\delta(0) > 0$, $\dot{delta}(0) > 0$, $P_e(0) > 0$. *(see Errata above)*

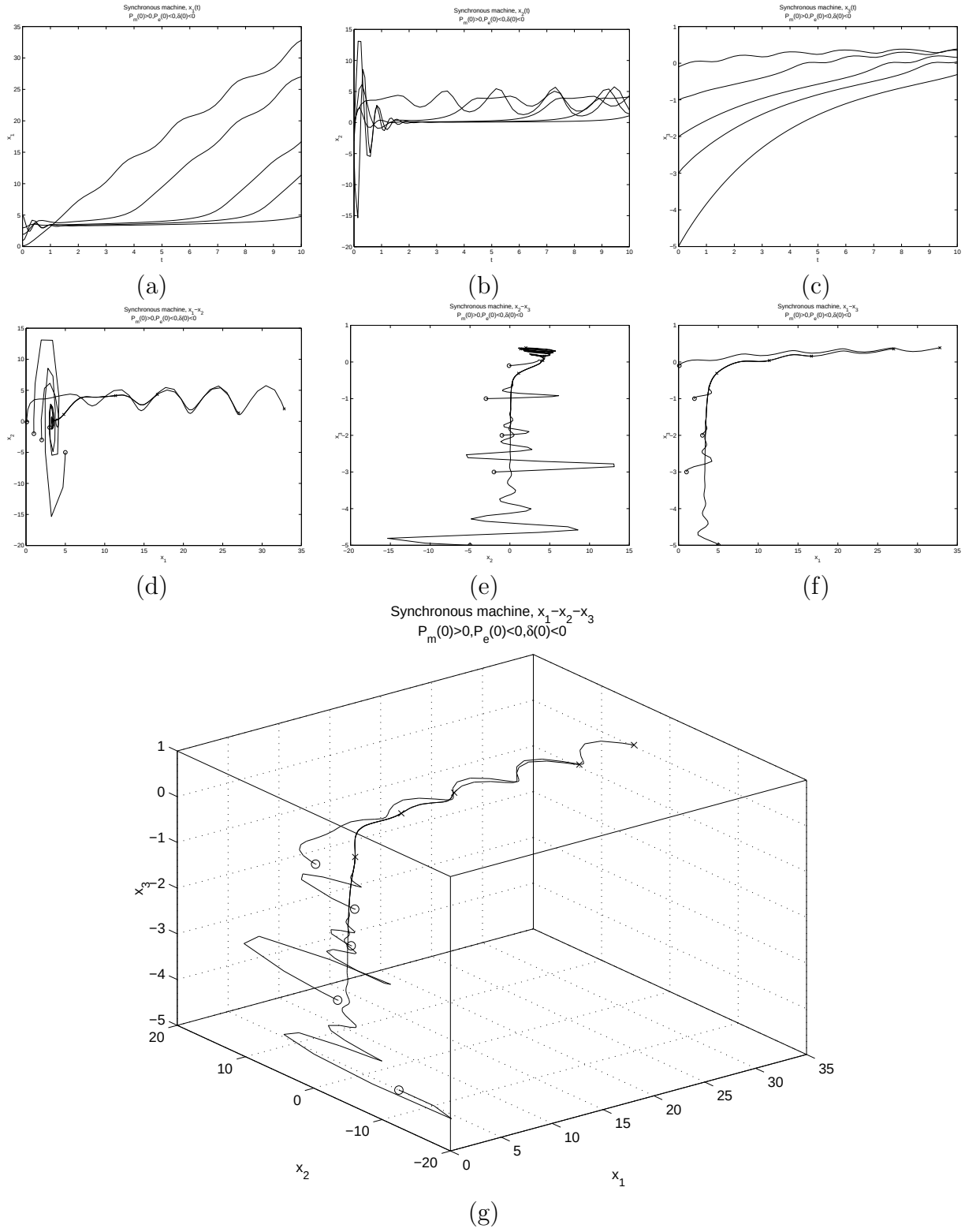
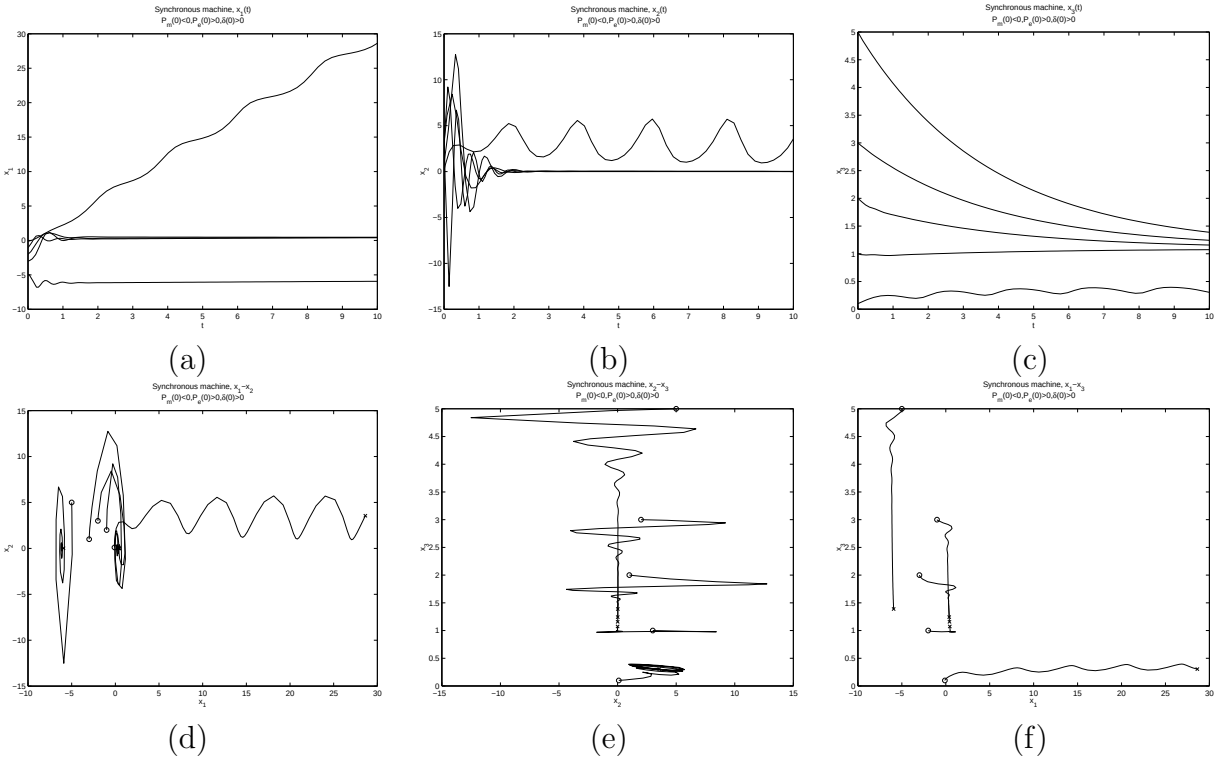
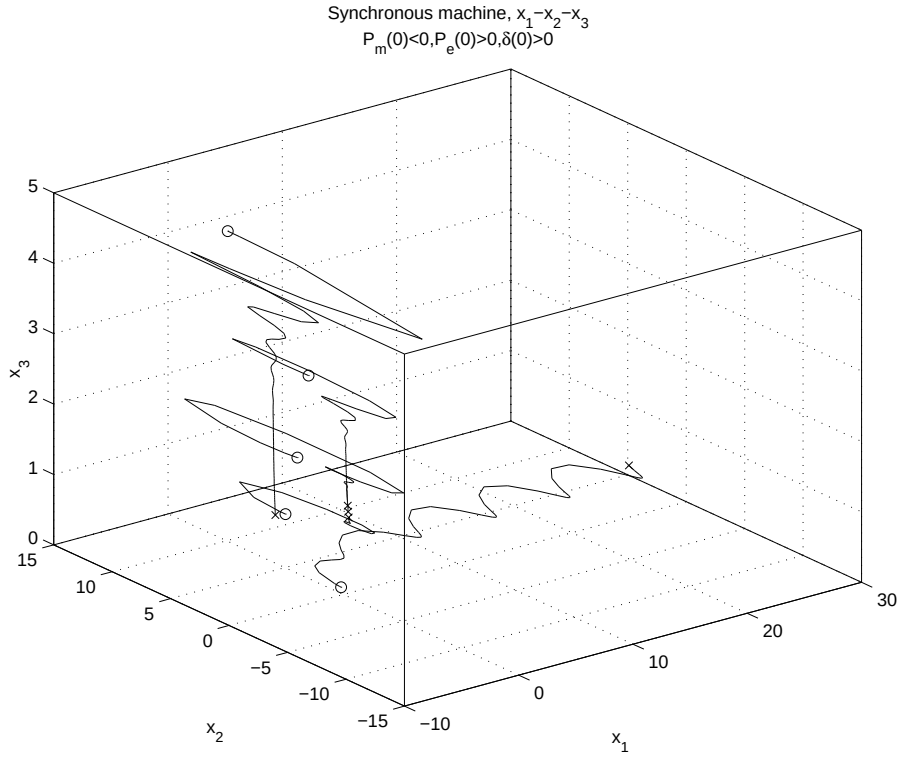


Figure 52: $\delta(0) > 0$, $\dot{delta}(0) > 0$, $P_e(0) > 0$. *(see Errata above)*



(a) (b) (c)

(d) (e) (f)



(g)

Figure 53: $\delta(0) > 0$, $\dot{delta}(0) > 0$, $P_e(0) > 0$. *(see Errata above)*

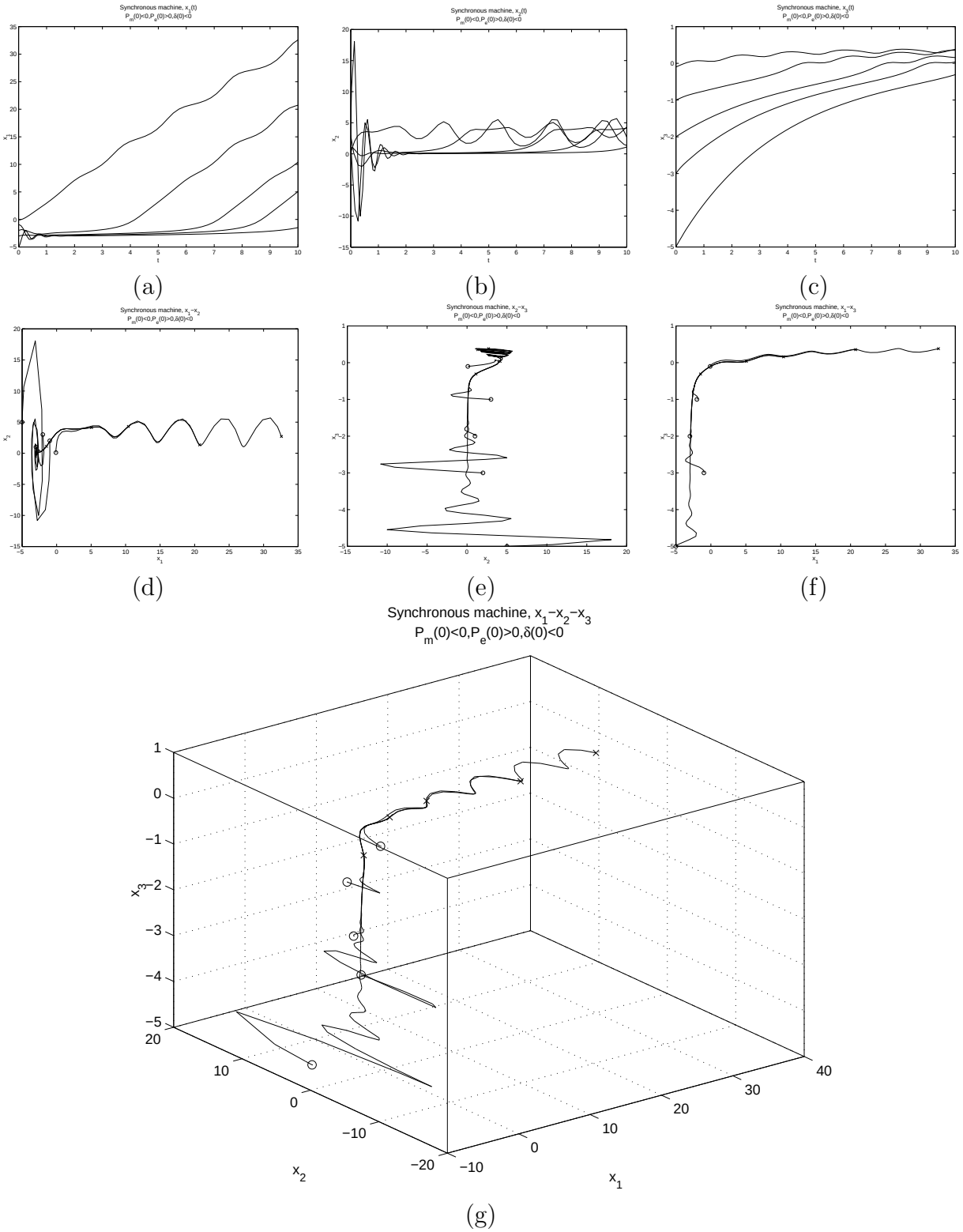
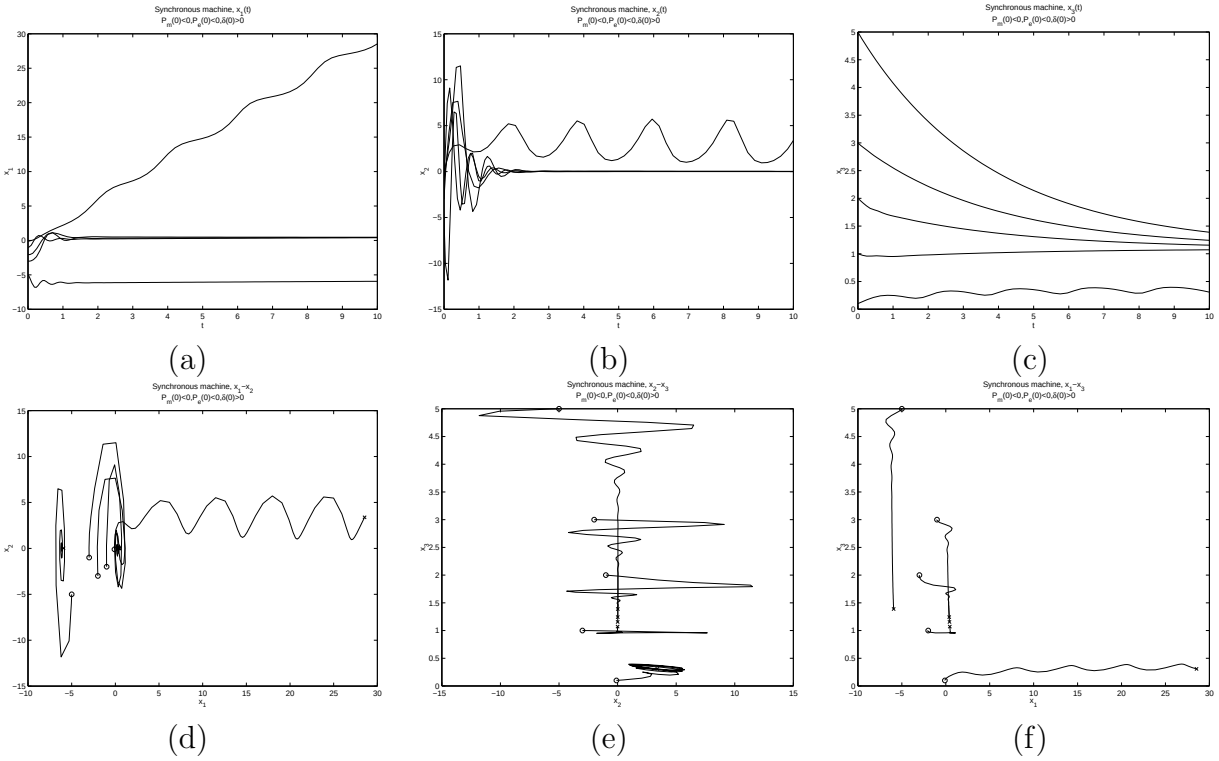
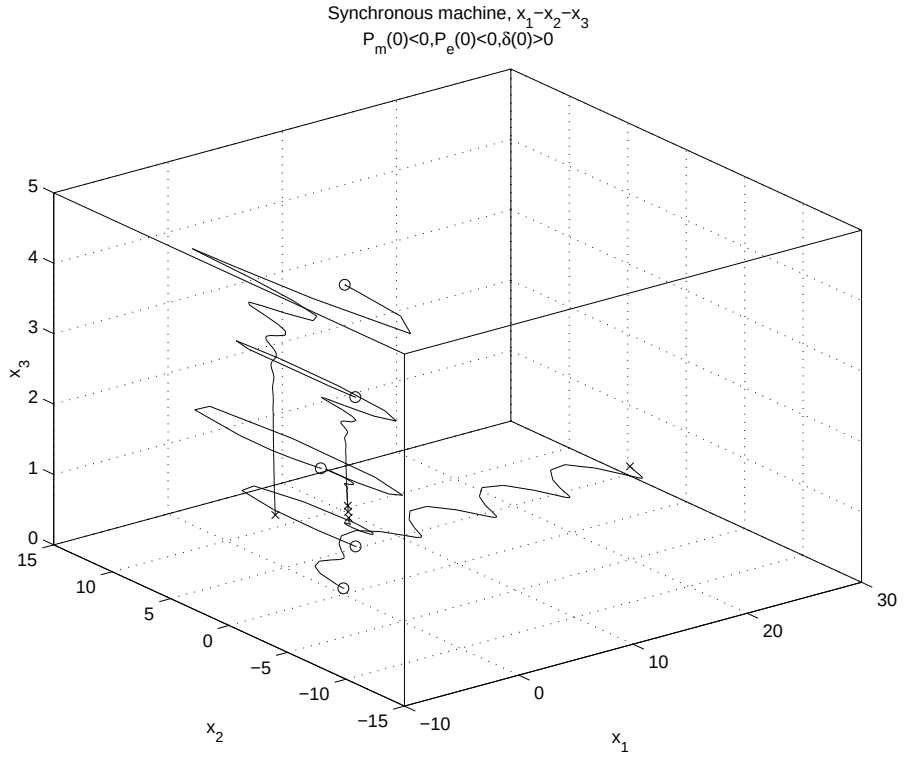


Figure 54: $\delta(0) > 0$, $\dot{delta}(0) > 0$, $P_e(0) > 0$. *(see Errata above)*



(a) (b) (c)

(d) (e) (f)



(g)

Figure 55: $\delta(0) > 0$, $\dot{delta}(0) > 0$, $P_e(0) > 0$. *(see Errata above)*

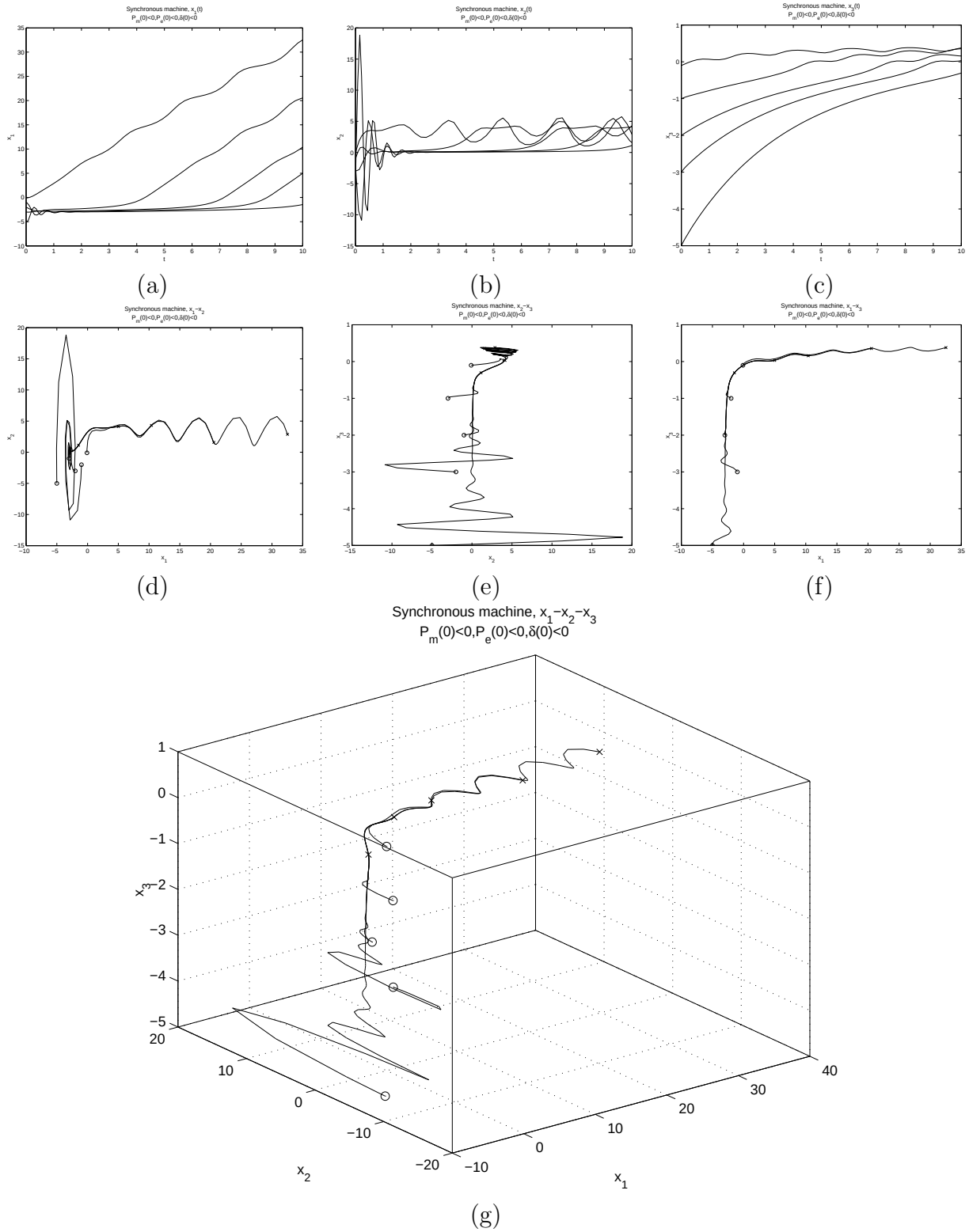


Figure 56: $\delta(0) > 0$, $\dot{delta}(0) > 0$, $P_e(0) > 0$. *(see Errata above)*

Exact feedback linearization technique

Let the system be described by

$$\dot{x} = f(x) + g(x)u. \tag{152}$$

By exact feedback linearization we mean [Kha96] transforming Equation 152 into

$$\dot{z} = Az + B\beta^{-1}(x)[u - \alpha(x)], \quad (A, B) \text{ controllable} \tag{153}$$

and then apply to the system an input

$$u = \alpha(x) + \beta(x)v. \tag{154}$$

Let

$$z = T(x), \tag{155}$$

then

$$\dot{z} = \frac{\partial T}{\partial x}\dot{x} = \frac{\partial T}{\partial x}\left[f(x) + g(x)u\right] \Longrightarrow AT(x) + B\beta^{-1}(x)[u - \alpha(x)]. \tag{156}$$

Therefore

$$\frac{\partial T}{\partial x}f(x) = AT(x) - B\beta^{-1}(x)\alpha(x) \tag{157}$$

$$\frac{\partial T}{\partial x}g(x) = B\beta^{-1}(x) \tag{158}$$

Then use a linear state transformation

$$\zeta = Mz \tag{159}$$

to transform the system into

$$\dot{\zeta} = MAM^{-1}\zeta + MB\beta^{-1}(x)[u - \alpha(x)]. \tag{160}$$

Choose M such that

$$MAM^{-1} = A_c + B_c\lambda^T, \quad MB = B_c,$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}_{n\times n}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n\times 1}$$

Then

$$A_cT(x) - B_c\beta^{-1}(x)\alpha(x) = \begin{bmatrix} T_2(x) \\ T_3(x) \\ \vdots \\ T_n(x) \\ -\frac{\alpha(x)}{\beta(x)} \end{bmatrix}, \quad B_c\beta^{-1}(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{\beta(x)} \end{bmatrix}, \quad T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \vdots \\ T_{n-1}(x) \\ T_n(x) \end{bmatrix}.$$

Therefore the method is to find $T_1(x)$ such that

$$\frac{\partial T_i}{\partial x}g(x) = 0, \quad i = 1, 2, \ldots, n-1; \quad \frac{\partial T_n}{\partial x}g(x) \neq 0, \quad T_{i+1} = \frac{\partial T_i}{\partial x}f(x), \tag{161}$$

which then gives

$$\beta(x) = \frac{1}{\left(\frac{\partial T_n}{\partial x}\right)g(x)}, \quad \alpha(x) = -\frac{\left(\frac{\partial T_n}{\partial x}f(x)\right)}{\frac{\partial T_n}{\partial x}g(x)}. \tag{162}$$

Also, choose the transformation T such that at the equilibrium point x_{eq}

$$T\left(x_{eq}\right) = 0$$

in order to map the equilibrium point into the origin in the new set of state equations.

Exact feedback linearization of the generator model

A third-order model of a synchronous generator connected to an infinite bus could be controlled by the *exact feedback linearization* ([Kha96], Example 12.2) Refering to Equation 152, the synchronous generator has

$$f(x) = \begin{bmatrix} x_2 \\ -a\left[(1+x_3)\sin\left(x_1+\delta\right) - \sin\delta\right] - bx_2 \\ -cx_3 + d\left[\cos\left(x_1+\delta\right)\right] \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

a, b, c, d and δ are positive constants. The conditions to be satisfied are

$$\frac{\partial T_1}{\partial x}g = 0, \quad \frac{\partial T_2}{\partial x}g = 0, \quad \frac{\partial T_3}{\partial x}g \neq 0, \quad T_2(x) = \frac{\partial T_1}{\partial x}f(x), \quad T_3(x) = \frac{\partial T_2}{\partial x}f(x), \quad T_1 = 0.$$

Referring the detailed calculation to the reference ([Kha96]) the control scheme can be summarized as follows,

$$\left.\begin{array}{ll} z_1 & = x_1 \\ z_2 & = x_2 \\ z_3 & = -a\left[(1+x_3)\sin\left(x_1+\delta\right) - \sin\delta\right] - bx_2 \end{array}\right\}, \tag{163}$$

where $z = T(x)$ is defined for

$$0 < x_1 + \delta < \pi.$$

And

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 \\ x_3 &= -1 - \frac{z_3 + bz_2 - a\sin\delta}{a\sin(z_1+\delta)} \end{aligned}$$

where $x = T^{-1}(z)$ is defined for

$$0 < z_1 + \delta < \pi.$$

The state-equation in the z-coordinate is

$$\left.\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= -a\sin(x_1+\delta)\,[u-\alpha(x)] \end{aligned}\right\}, \tag{164}$$

and the control input is

$$u = \alpha(x) - \frac{1}{a\sin(x_1+\delta)}v, \tag{165}$$

where

$$\alpha(x) = \frac{-a(1+x_3)\cos(x_1+\delta)\,f_1(x) - bf_2(x) - a\sin(x_1+\delta)\,f_3(x)}{a\sin(x_1+\delta)} \tag{166}$$

$$\beta(x) = -\frac{1}{a\sin(x_1+\delta)}. \tag{167}$$

Simulation of realized system using exact feedback linearization

Let $a = 1.3$, $b = 2.25$, $c = 1.7$, $d = 1.14$ and $\delta = 0.79$, then

$$\begin{aligned} \dot{x}_1 &= x_2 && (168)\\ \dot{x}_2 &= -1.3\left[(1+x_3)\sin(x_1+0.79) - \sin 0.79\right] - 2.25x_2 && (169)\\ \dot{x}_3 &= -1.7x_3 + 1.14\left[\cos(x_1+0.79)\right] + u. && (170) \end{aligned}$$

$$\left.\begin{aligned} z_1 &= x_1\\ z_2 &= x_2\\ z_3 &= -1.3\left[(1+x_3)\sin(x_1+0.79) - \sin 0.79\right] - 2.25x_2 \end{aligned}\right\}, \quad (171)$$

where $z = T(x)$ is defined for

$$0 < x_1 + 0.79 < \pi \quad \Longrightarrow \quad -0.79 < x_1 < 2.35.$$

And

$$\begin{aligned} x_1 &= z_1\\ x_2 &= z_2\\ x_3 &= -1 - \frac{z_3 + 2.25z_2 - 1.3\sin 0.79}{1.3\sin(z_1+0.79)} \end{aligned}$$

where $x = T^{-1}(z)$ is defined for

$$0 < z_1 + 0.79 < \pi \quad \Longrightarrow \quad -0.79 < z_1 < 2.35.$$

The state-equation in the z-coordinate is

$$\left.\begin{aligned} \dot{z}_1 &= z_2\\ \dot{z}_2 &= z_3\\ \dot{z}_3 &= -1.3\sin(x_1+0.79)\left[u - \alpha(x)\right] \end{aligned}\right\}, \quad (172)$$

and the control input is

$$u = \alpha(x) - \frac{1}{1.3\sin(x_1+0.79)}v, \quad (173)$$

where

$$\alpha(x) = \frac{-1.3\,(1+x_3)\cos(x_1+0.79)\,f_1(x) - 2.25f_2(x) - 1.3\sin(x_1+0.79)\,f_3(x)}{1.3\sin(x_1+0.79)} \quad (174)$$

$$\beta(x) = -\frac{1}{1.3\sin(x_1+0.79)} \quad (175)$$

Next is the simulation result done using this technique. Figure 57 was simulated with initial conditions

$$(z_1, z_2, z_3) = (0, 0, -2), (0, 0, 0), (0, 0, 2), (0, 0, 4),$$

which correspond to

$$(x_1, x_2, x_3) = (0, 0, 2.1658), (0, 0, 0), (0, 0, -2.1658), (0, 0, -4.3315)$$

respectively.
Figure 57 shows that this technique can be used to produce a nice control when the initial conditions are within the domain of attraction. Not only the initial conditions should satisfy Equation and , but also during the course of simulation the state variables must not at anytime violate these two equations.

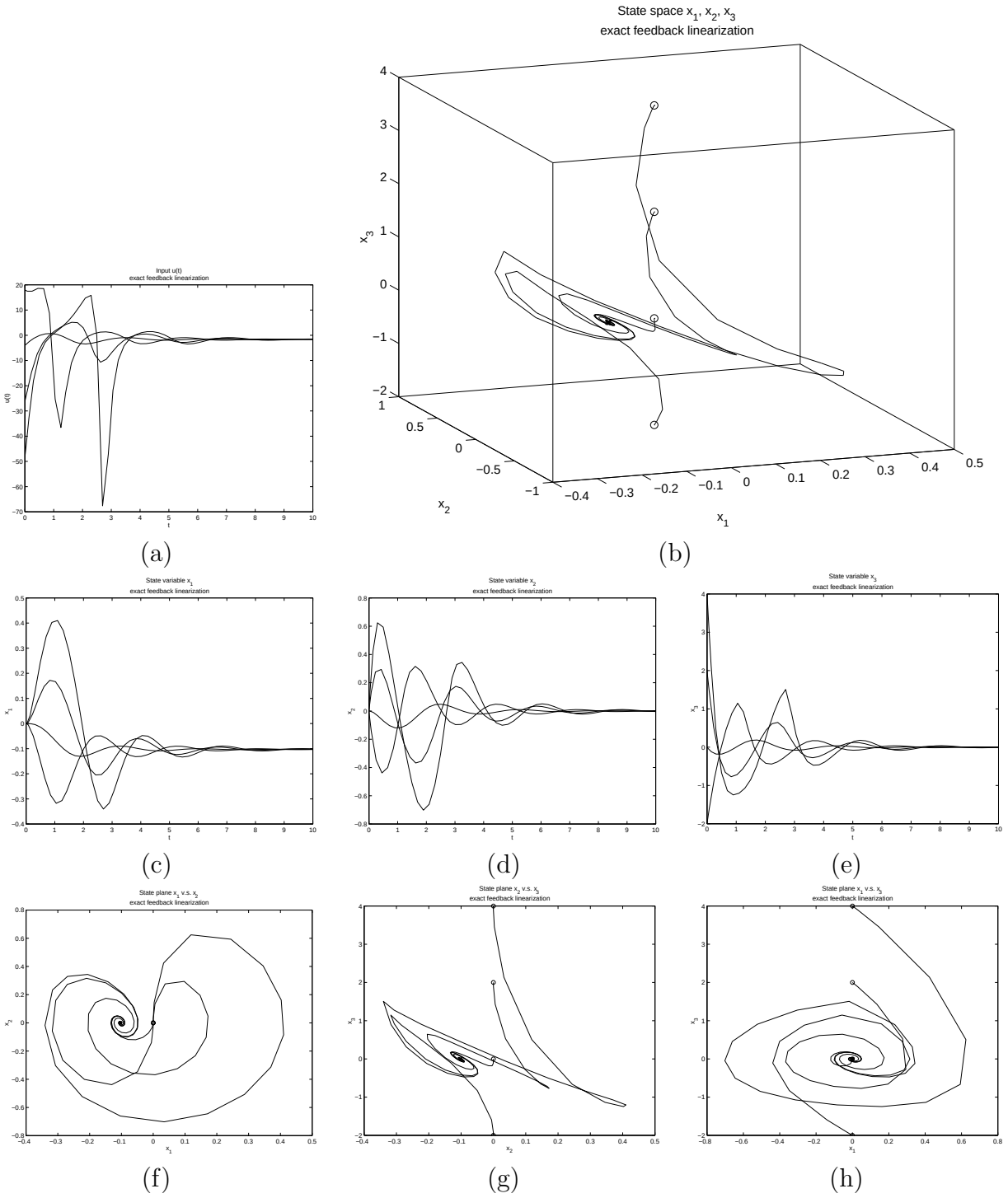


Figure 57: *Exact feed back linearization technique on a synchronous machine. Initial states* (x_1, x_2, x_3) *were at* $(0, 0, 2.1658)$, $(0, 0, 0)$, $(0, 0, -2.1658)$, $(0, 0, -4.3315)$

Figure 58 shows the result of the controlled system at the verge of instability. Here the initial conditions were

$$(z_1, z_2, z_3) = (-0.5, 0.5, -0.5),$$

which corresponds to

$$(x_1, x_2, x_3) = (-0.5000, 0.5000, -0.1971).$$

Looking at Figure 58 one can see that the system was finally stabilized at the final time $t = 10$ seconds as shown in Figure 58 (b). But during the course of control, at $t \approx 0.7$ seconds, the state $x_3 \to 1.9 \times 10^{13}$ (Figure 58 (e)) and the input $u \to 0.85 \times 10^{28}$ (Figure 58 (a)). Therefore this control scheme is not useful in practice.

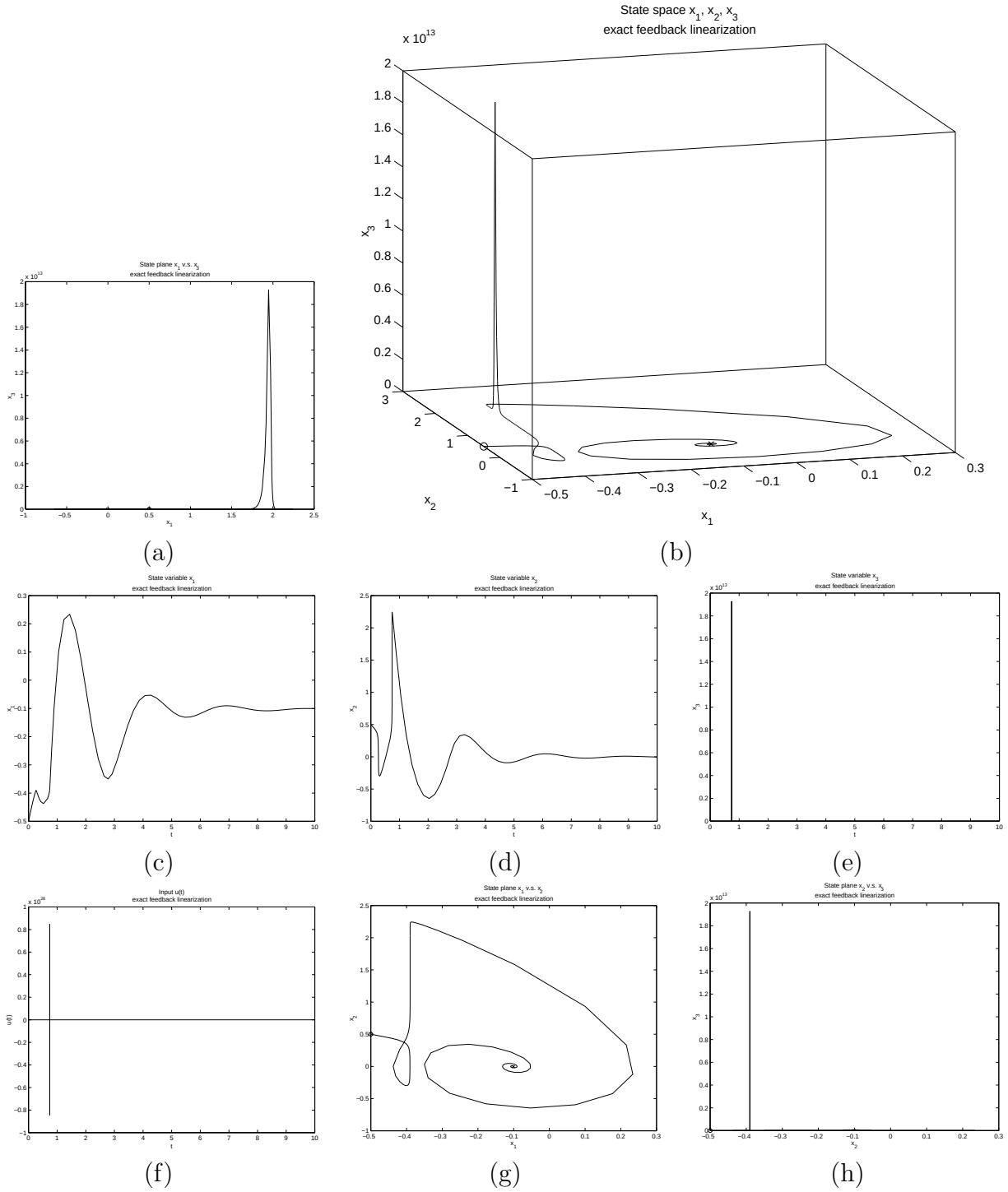


Figure 58: *At the verge of instability, exact feed back linearization technique on a synchronous machine. Initial states* $(x_1, x_2, x_3) = (-0.5000, 0.5000, -0.1971)$

Figure 59 shows another result when the controlled system was close to instability. Here the initial conditions were

$$(z_1, z_2, z_3) = (-0.3, 1.9, 0),$$

which corresponds to

$$(x_1, x_2, x_3) = (-0.3000, 1.9000, -6.4780).$$

As was the case with Figure 58, in Figure 59 the system was finally stabilized at the final time $t = 10$ seconds as shown in Figure 59 (b). At $t \approx 2.2$ seconds when the state $x_3 \to 9$ (Figure 59 (e)), the input $u \to -13.5 \times 10^4$ (Figure 59 (a)).

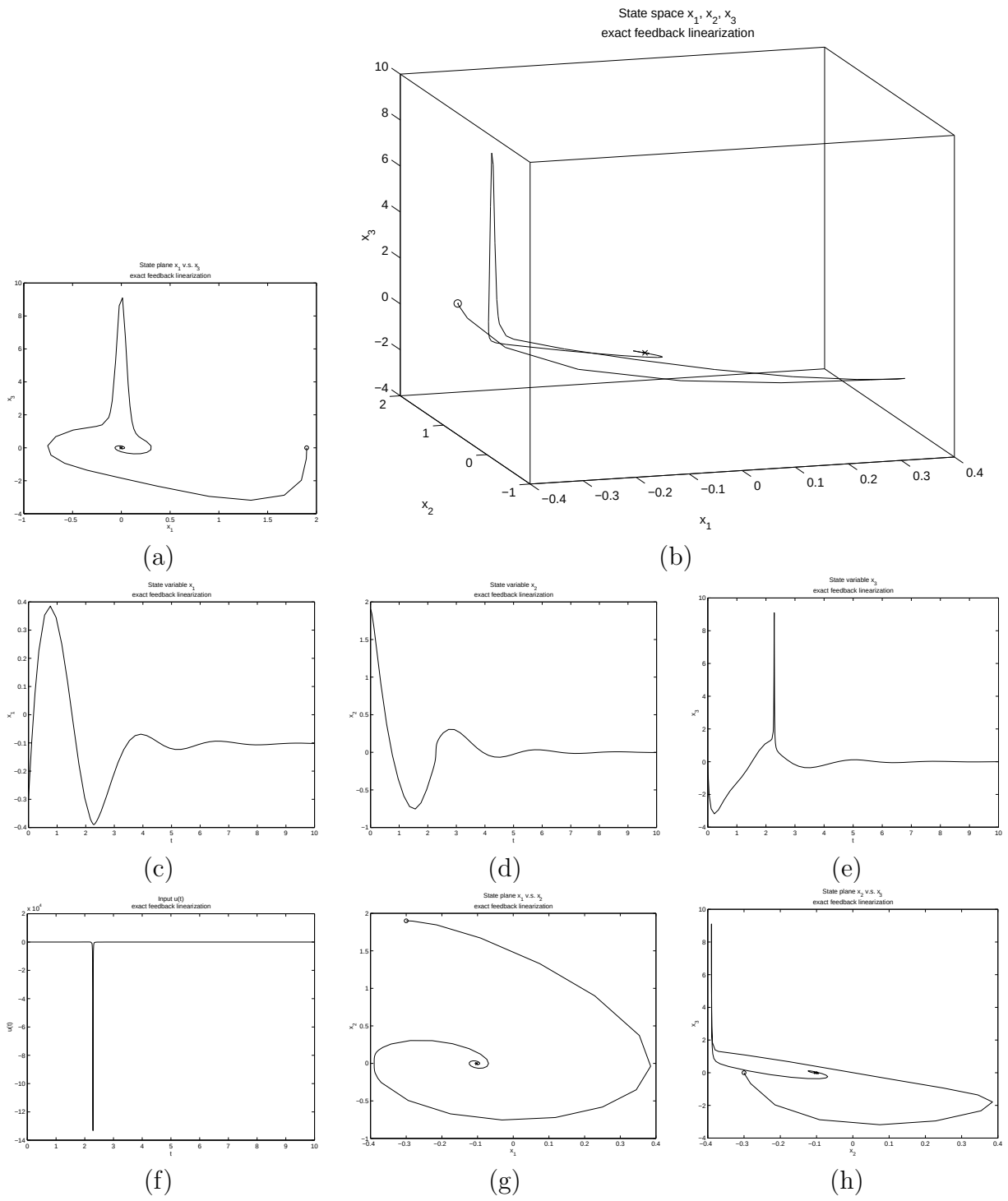


Figure 59: *At the verge of instability, exact feed back linearization technique on a synchronous machine. Initial states* $(x_1, x_2, x_3) = (-0.3000, 1.9000, -6.4780)$

Domain of stability

The domain of stability is the domain of initial states $(x_1(0), x_2(0), x_3(0))$ from which the asymptotical stability lf all trajectories starting from within it is guaranteed, ie

$$\lim_{t\to\infty}(x_1(t), x_2(t), x_3(t)) \Leftrightarrow (x_1(0), x_2(0), x_3(0)) \in \mathcal{D}_s,$$

where $\mathcal{D}_s$ represents the domain if stability. To give an idea about the shape of $\mathcal{D}_s$ for the system currently under consideration, Figure 60 shows the plot obtained from simulations done with initial states

$x_2 = 0$, $(x_1(0), x_2(0), x_3(0))$ $=$
$(0,0,-2)$, $(0,0,0)$, $(0,0,2)$, $(0,0,4)$, $(.5,0,0)$, $(-.3,0,0)$, $(-.3,0,.5)$, $(-.3,0,.7)$,
$(-.3,0,3)$, $(-.3,0,7)$, $(-.3,0,9)$, $(-.3,0,9.9)$, $(-.3,0,-1)$, $(-.3,0,-1.3)$,
$(.5,0,.9)$, $(.5,0,.7)$, $(.5,0,-.5)$, $(.5,0,.5)$, $(.5,0,-.7)$, $(.5,0,-.8)$ (Figure 60 (e)),

$x_1 = -0.3$, $(x_1(0), x_2(0), x_3(0)) = (-.3,.5,.5)$, $(-.3,.5,-.5)$, $(-.3,0,0)$, $(-.3,0,.5)$,
$(-.3,0,.7)$,
$(-.3,0,3)$, $(-.3,0,7)$, $(-.3,0,9)$, $(-.3,0,9.9)$, $(-.3,0,-1)$, $(-.3,0,-1.3)$,
$(-.3,-.2,0)$, $(-.3,-.25,0)$, $(-.3,1,0)$, $(-.3,1.7,0)$, $(-.3,1.8,0)$, $(-.3,.3,.3)$,
$(-.3,-.2,-.2)$, $(-.3,-.3,.3)$, $(-.3,.3,-.3)$, $(-.3,-.3,.4)$, $(-.3,-.3,.6)$,
$(-.3,-.3,2)$, $(-.3,-.3,4)$, $(-.3,-.3,6)$, $(-.3,-.3,10)$, $(-.3,-.3,10.5)$,
$(-.3,.5,7)$, $(-.3,.5,8)$, $(-.3,1,5.7)$, $(-.3,-.4,4)$, $(-.3,-.5,6)$, $(-.3,-.6,5)$,
$(-.3,-.7,2)$, $(-.3,-.7,7)$, $(-.3,-.6,9)$,
$(-.3,-.6,10)$, $(-.3,-.7,10)$, $(-.3,-.8,10)$, $(-.3,-1,10)$, $(-.3,-2,10)$,
$(-.3,-1.5,7)$, $(-.3,-1.5,11)$, $(-.3,-1.5,12)$, $(-.3,-1,12)$, $(-.3,1,-1)$,
$(-.3,1,-2)$, $(-.3,1,-3)$, $(-.3,2,-4)$, $(-.3,2,-5)$, $(-.3,2,-6)$, $(-.3,2,-7)$,
$(-.3,1,-6)$, $(-.3,1.5,-7)$, $(-.3,1.5,3)$ (Figure 60 (b)),

$x_1 = 0.5$, $(x_1(0), x_2(0), x_3(0)) = (.5,-.5,.5)$, $(.5,-.5,-.5)$, $(.5,0,0)$, $(.5,-.7,0)$,
$(.5,-.5,0)$, $(.5,0,.9)$, $(.5,0,.7)$, $(.5,0,-.5)$, $(.5,0,.5)$, $(.5,0,-.7)$, $(.5,0,-.8)$,
$(.5,-2.3,9)$, $(.5,-2.3,8)$, $(.5,-2.3,6)$, $(.5,-2.3,5)$, $(.5,-2.3,4)$, $(.5,-2,4)$,
$(.5,-1.5,4)$, $(.5,-.9,4)$, $(.5,-.7,4)$, $(.5,-.7,3)$, $(.5,-.7,2)$, $(.5,-.7,1)$,
$(.5,-3,6)$, $(.5,-2.5,7)$, $(.5,-4,10)$, $(.5,-4,11)$, $(.5,-3,11)$, $(.5,-3,10)$,
$(.5,-3,9)$, $(.5,-3,8)$, $(.5,-4,9)$ (Figure 60 (c)),

$x_1 = 1$, $(x_1(0), x_2(0), x_3(0)) = (1,-2.5,7)$, $(1,-2.2,7)$,
$(1,-2,8)$, $(1,-2,7)$, $(1,-2,6)$, $(1,-5,16)$, $(1,-5,15)$, $(1,-5,14)$, $(1,-4,14)$,
$(1,-4,13)$, $(1,-3.8,13)$, $(1,-3.8,12)$, $(1,-3.8,11)$, $(1,-3.8,10)$, $(1,-3.5,10)$,
$(1,-2.9,10)$, $(1,-2.5,10)$, $(1,-2.5,9)$, $(1,-2.5,8)$, $(1,-2.8,7.8)$, $(1,-2.3,7.8)$,
$(1,-3,7.8)$, $(1,-3,9)$, $(1,-3,8)$. (Figure 60 (d))

Figure 60 give us a rough picture how the domain of attraction for this system is a tubular strip running roughly along the alignment of x_1-axis. The location of the points were chosen by trial and error. The figure also gives convex hulls for each group of data plotted.

The domain of attraction shown is not large, therefore this kind of control scheme may not be very robust to the value of the state variables.

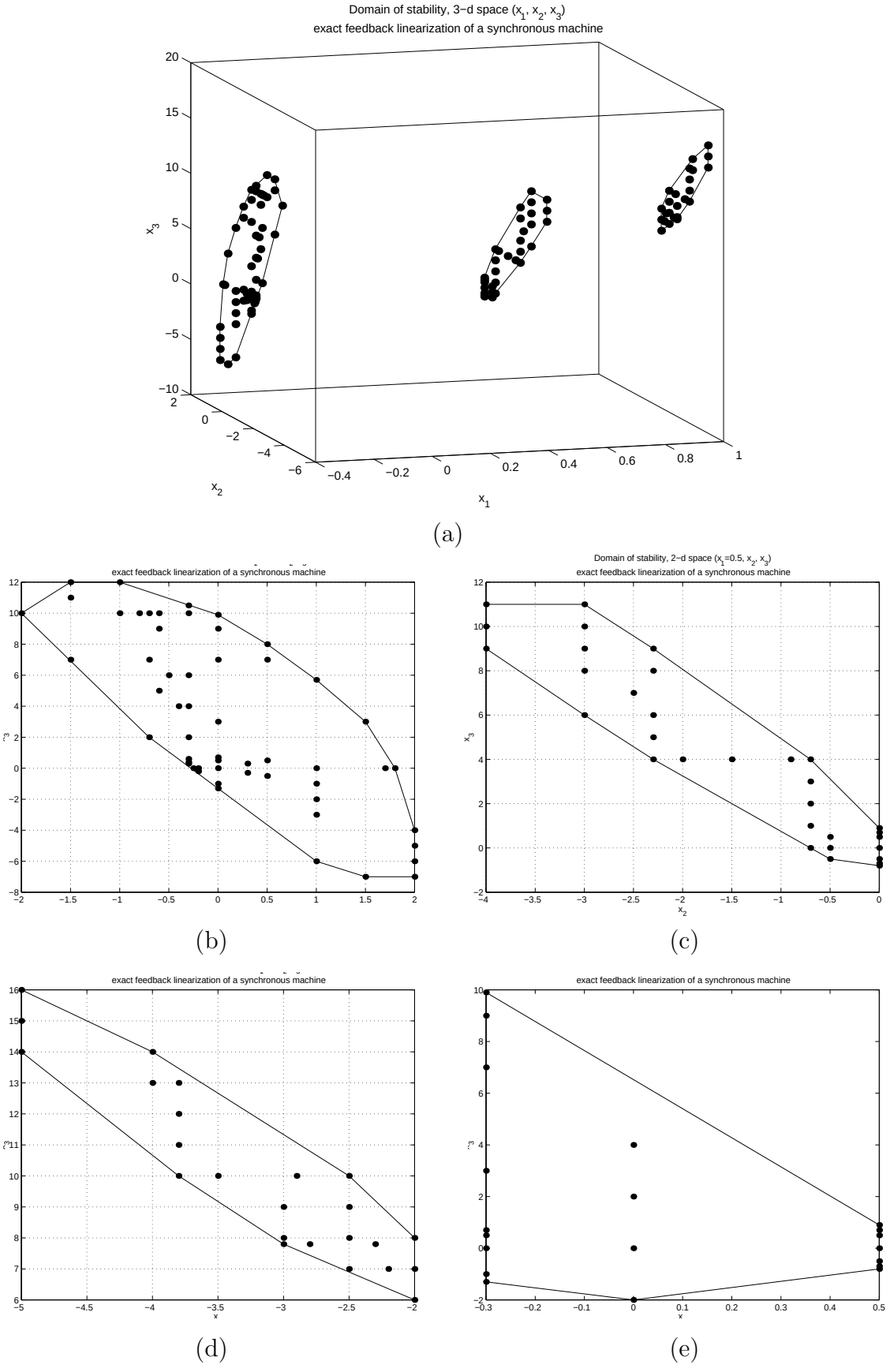


Figure 60: *Domain of stability, cross sections. (b) cross section in* 2*-d along* $x_1 = -0.3$ *slice, (c) along* $x_1 = 0.5$ *slice, (d) along* $x_1 = 1$ *slice, (e) along* $x_2 = 0$ *slice, and (a) cross section in* 3*-d of the slices shown in (b) upto (d)*

Tracking control of the generator model

For the synchronous generator described by Equation 143 to 145, suppose that the objective of control is to make the trajectory follow the curve of a function $\mathcal{F}$ by controlling the field voltage. Let

$$\mathcal{F} = x_1^2 + x_2^2 + x_3^2 - r^2. \tag{176}$$

Let the state equations of the system be

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= a_1 + a_2 x_2 + a_3 x_3 \sin\, x_1 \\ \dot{x_3} &= a_4 x_3 + a_5 \cos\, x_1 + a_6 E_{FD}, \end{aligned}$$

where $a_1 = \frac{P_m}{H}$, $a_2 = -\frac{D}{H}$, $a_3 = -\frac{\eta_1}{H}$, $a_4 = -\frac{\eta_2}{\tau}$, $a_5 = \frac{\eta_3}{\tau}$ and $a_6 = \frac{1}{\tau}$.
Let the Lyapunov function be

$$V = \mathcal{F}^2 = \left(x_1^2 + x_2^2 + x_3^2 - r^2\right)^2. \tag{177}$$

Than the derivative is

$$\begin{aligned} \dot{V} &= (\dot{\mathcal{F}^2}) = 2\mathcal{F}\dot{\mathcal{F}} \\ &= 2\mathcal{F}\mathcal{F}_x^T \dot{x} \\ &= 2\left(x_1^2 + x_2^2 + x_3^2 - r^2\right)(2x_1\dot{x}_1 + 2x_2\dot{x}_2 + 2x_3\dot{x}_3) \\ &= 4\left(x_1^2 + x_2^2 + x_3^2 - r^2\right)(x_1 x_2 + x_2\,(a_1 + a2x_2 + a_3 x_3 \sin x_1) + \\ &\qquad x_3\,(a_4 + a_5 \cos x_1 + a_6)). \end{aligned} \tag{178}$$

Let the input be

$$u = -K\mathrm{sgn}\left(g\mathcal{F}\mathcal{F}_x^T\mathcal{V}\right),$$

where $\mathcal{V}$ is a unit vector and $g = a_6 = \frac{1}{\tau}$, then

$$E_{FD} = u = -K\mathrm{sgn}\left(a_6\left(x_1^2 + x_2^2 + x_3^2 - r^2\right)\begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \tag{179}$$

$$= -K\mathrm{sgn}\left(2a_6\left(x_1^2 + x_2^2 + x_3^2 - r^2\right)(x_1 + x_2 + x_3)\right) \tag{180}$$

Figure 61 to 65 shows the result from simulation when parameters are

$$P_m = 0.9;\ H = 0.02;\ \eta_1 = 2.2;\ \eta_2 = 2.5;\ \eta_3 = 1.9;\ \tau = 7;\ r = 2.5;\ K = -3.2;\ \frac{D}{H} = 5,$$

which, when recall that

$$a_1 = \frac{P_m}{H};\quad a_2 = \frac{D}{H};\quad a_3 = -\frac{\eta_1}{H};\quad a_4 = -\frac{\eta_2}{\tau};\quad a_5 = \frac{\eta_3}{\tau};\quad a_6 = \frac{1}{\tau},$$

results in the model's parameters of

$$a_1 = 45; \quad a_2 = -5; \quad a_3 = -110; \quad a_4 = -0.3571; \quad a_5 = 0.2714; \quad a_6 = 0.1429.$$

Here $\circ$ represents a starting point while $\triangle$ represents an end point. Initial conditions of the states were

Figure 61 (a) $(3,3,3)$, (b) $(-1,-1,-1)$,

Figure 62 (a) $(1,1,1)$, $(2,2,2)$, $(3,3,3)$, and (b) $(1,1,-1)$, $(2,2,-2)$, $(3,3,-3)$,

Figure 63 (a) $(1,-1,1)$, $(2,-2,2)$, $(3,-3,3)$, and (b) $(-1,1,1)$, $(-2,2,2)$, $(-3,3,3)$,

Figure 64 (a) $(-1,-1,-1)$, $(-2,-2,-2)$, $(-3,-3,-3)$, and (b) $(-1,-1,1)$, $(-2,-2,2)$, $(-3,-3,3)$,

Figure 65 (a) $(-1,1,-1)$, $(-2,2,-2)$, $(-3,3,-3)$, and (b) $(1,-1,-1)$, $(2,-2,-2)$, $(3,-3,-3)$,

The input used was

$$u = -3.2\mathrm{sgn}\left(2a_6\left(x_1^2 + x_2^2 + x_3^2 - r^2\right)(x_1 + x_2 + x_3)\right) \tag{181}$$

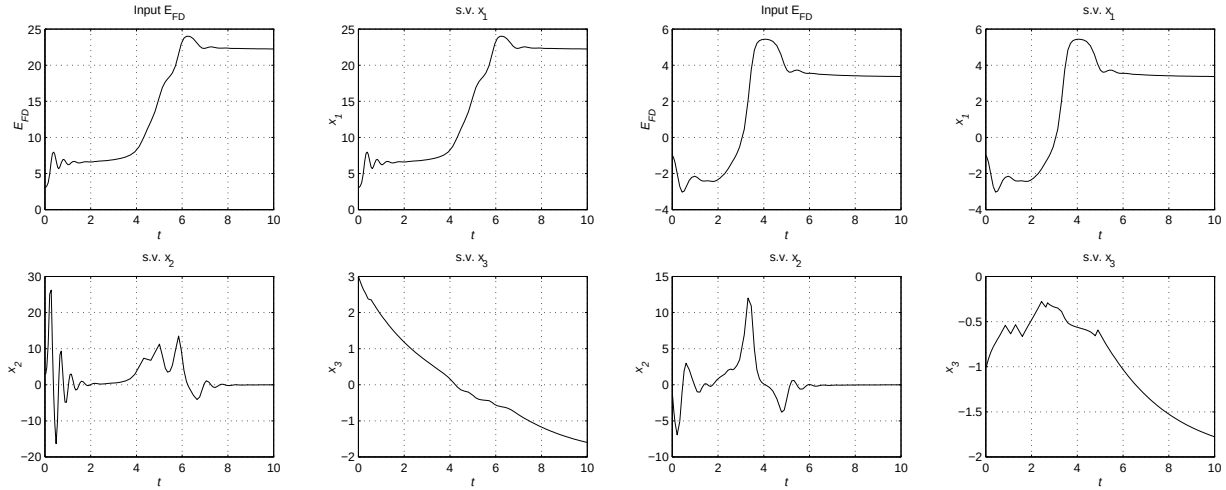
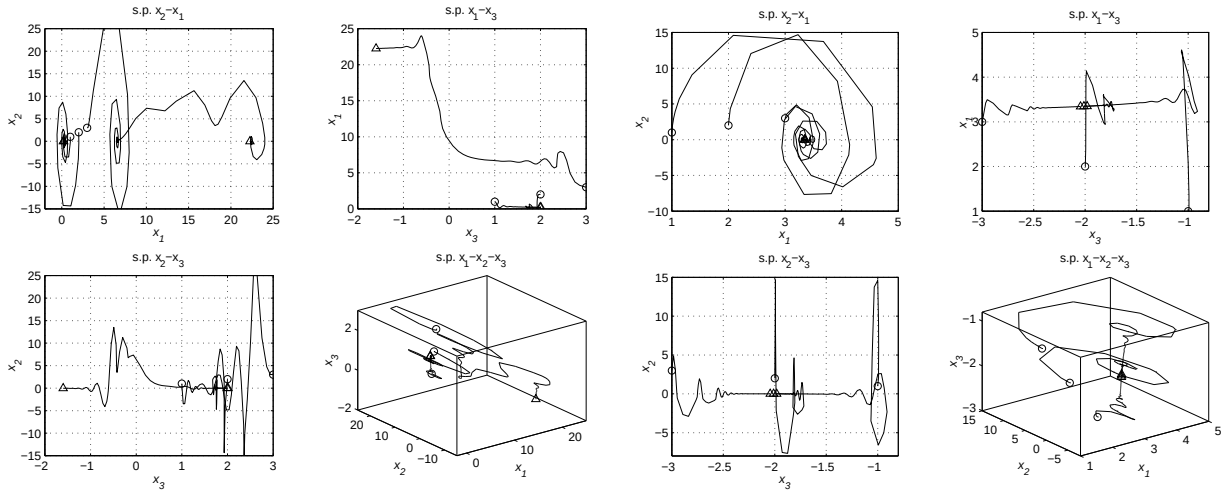
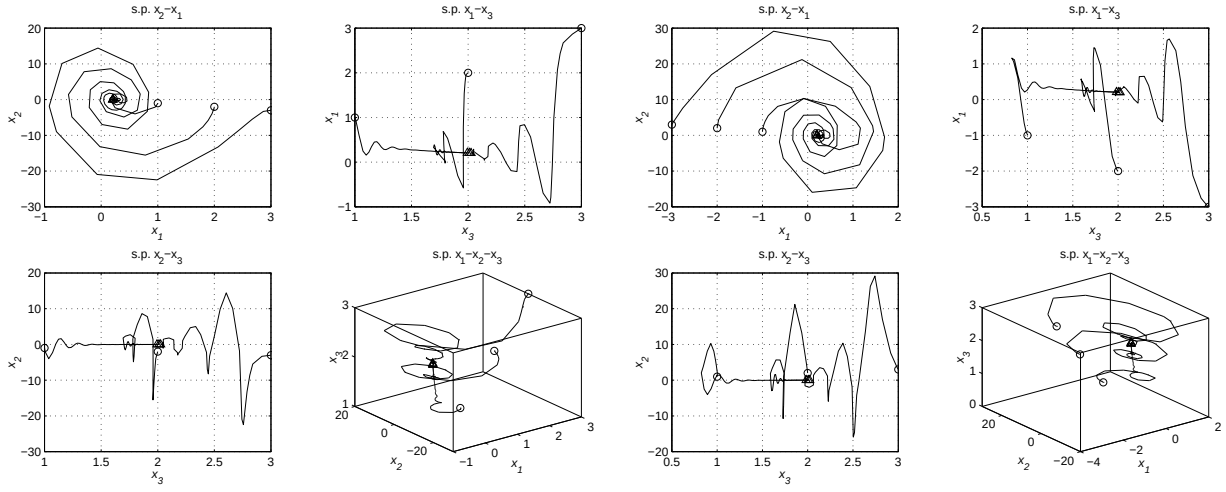


Figure 61: *State variables of, and input to the synchronous generator.* $u = -3.2\mathrm{sgn}\left(2a_6\left(x_1^2 + x_2^2 + x_3^2 - r^2\right)\ (x_1 + x_2 + x_3)\right)$



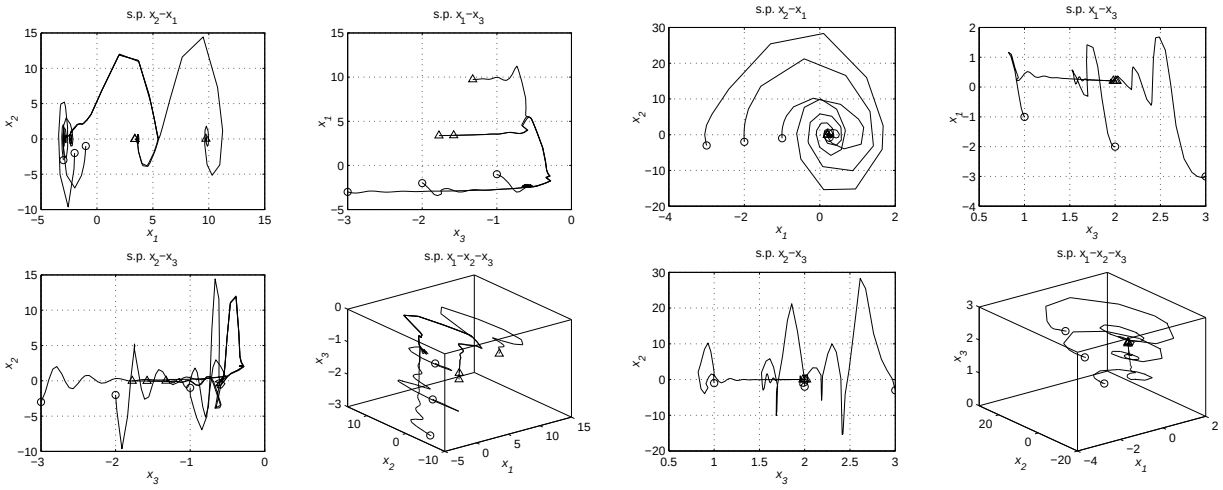
(a) (b)

Figure 62: *State planes of the synchronous generator.* $u = -3.2\mathrm{sgn}\left(2a_6\left(x_1^2+x_2^2+x_3^2-r^2\right)\left(x_1+x_2+x_3\right)\right)$



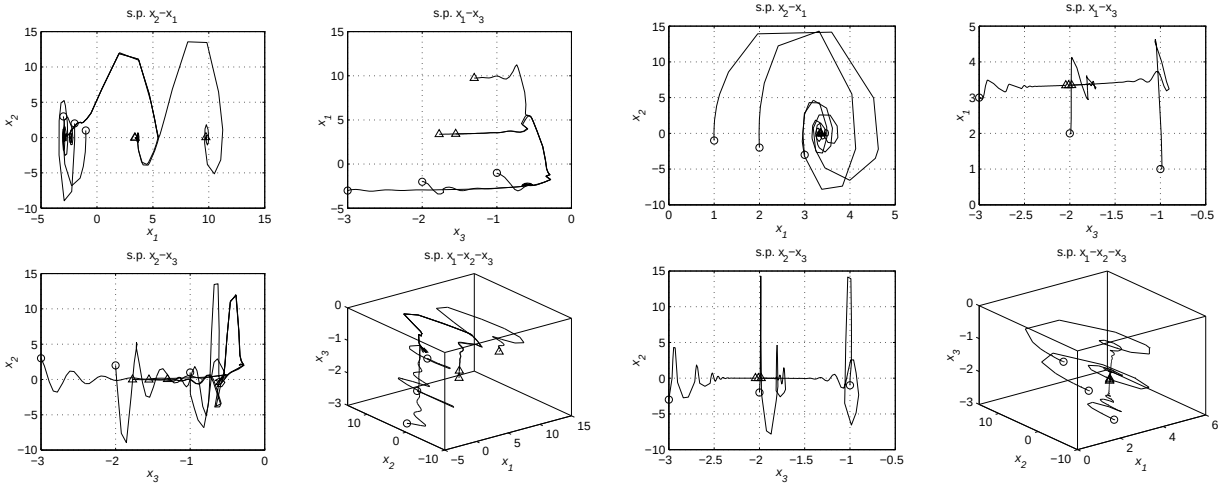
(a) (b)

Figure 63: *State planes of the synchronous generator.* $u = -3.2\mathrm{sgn}\left(2a_6\left(x_1^2+x_2^2+x_3^2-r^2\right)\left(x_1+x_2+x_3\right)\right)$



(a) (b)

Figure 64: *State planes of the synchronous generator.* $u = -3.2\mathrm{sgn}\left(2a_6\left(x_1^2+x_2^2+x_3^2-r^2\right)\left(x_1+x_2+x_3\right)\right)$



(a) (b)

Figure 65: *State planes of the synchronous generator.* $u = -3.2\mathrm{sgn}\left(2a_6\left(x_1^2+x_2^2+x_3^2-r^2\right)\left(x_1+x_2+x_3\right)\right)$

Synchronous machine, a seven state variables model

A synchronous machine can be represented by

$$\dot{x} = f(x, u, t) \tag{182}$$

which is actually [AF77]

$$\left[\begin{array}{c} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_Q \\ \hline \\ \dot{\omega} \\ \dot{\delta} \end{array}\right] = \left[\begin{array}{ccccc|cc} & & -L^{-1}\left(R+\omega N\right) & & & 0 & \\ \hline -\frac{L_d i_q}{3\tau_j} & -\frac{kM_F i_q}{3\tau_j} & -\frac{kM_D i_q}{3\tau_j} & \frac{L_d i_q}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & -\frac{D}{3\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{c} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ \hline \\ \omega \\ \delta \end{array}\right] + \left[\begin{array}{c} -L^{-1}v \\ \\ \hline \\ \frac{T_m}{3\tau_j} \\ -1 \end{array}\right], \tag{183}$$

in other words

$$v = -\left(R+\omega N\right) i - L\dot{i}, \tag{184}$$

where

$$\left(R+\omega N\right) = \left[\begin{array}{ccc|cc} r & 0 & 0 & \omega L_q & \omega k M_Q \\ 0 & r_F & 0 & 0 & 0 \\ 0 & 0 & r_D & 0 & 0 \\ \hline \\ -\omega L_d & -\omega k M_F & -\omega k M_D & r & 0 \\ 0 & 0 & 0 & 0 & r_Q \end{array}\right] \tag{185}$$

and

$$L = \left[\begin{array}{ccc|cc} L_d & kM_F & kM_D & 0 & 0 \\ kM_F & L_F & M_R & 0 & 0 \\ kM_D & M_R & L_D & 0 & 0 \\ \hline \\ 0 & 0 & 0 & L_q & kM_Q \\ 0 & 0 & 0 & kM_Q & L_Q \end{array}\right]. \tag{186}$$

Here i_d and i_q are d- and the q-axis components of stator currents and are related to the stator phase currents i_a, i_b and i_c by a Park's transformation equation

$$i_{0dq} \equiv P i_{abc}, \tag{187}$$

where the current vectors are

$$i_{0dq} = \left[\begin{array}{c} i_0 \\ i_d \\ i_q \end{array}\right], \quad i_{abc} = \left[\begin{array}{c} i_a \\ i_b \\ i_c \end{array}\right] \tag{188}$$

and the Park's transformation P is

$$P = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) \\ \sin\theta & \sin\left(\theta - \frac{2\pi}{3}\right) & \sin\left(\theta + \frac{2\pi}{3}\right) \end{bmatrix}. \tag{189}$$

The d-axis of the rotor is located at

$$\theta = \omega_R t + \delta + \frac{\pi}{2}, \tag{190}$$

where ω_R is the rated (synchronous) angular frequency and δ the synchronous torque angle.
The flux linkage equation of six mutually coupled coils, namely the three phase windings $sa - fa$, $sb - fb$, $sc - fc$ (here s and f stand for *start* and *finish* respectively), the field winding $F - F'$, the two damper windings $D - D'$ and $Q - Q'$, is

$$\left[\begin{array}{c} \lambda_a \\ \lambda_b \\ \lambda_c \\ \hline \lambda_F \\ \lambda_D \\ \lambda_Q \end{array}\right] = \left[\begin{array}{ccc|ccc} L_{aa} & L_{ab} & L_{ac} & L_{aF} & L_{aD} & L_{aQ} \\ L_{ba} & L_{bb} & L_{bc} & L_{bF} & L_{bD} & L_{bQ} \\ L_{aa} & L_{cb} & L_{cc} & L_{cF} & L_{cD} & L_{cQ} \\ \hline L_{Fa} & L_{Fb} & L_{Fc} & L_{FF} & L_{FD} & L_{FQ} \\ L_{Da} & L_{Db} & L_{Dc} & L_{DF} & L_{DD} & L_{DQ} \\ L_{Qa} & L_{Qb} & L_{Qc} & L_{QF} & L_{QD} & L_{QQ} \end{array}\right] \left[\begin{array}{c} i_a \\ i_b \\ i_c \\ \hline i_F \\ i_D \\ i_Q \end{array}\right] \quad \text{Wb turns}, \tag{191}$$

where

$$L_{aa} = L_s + L_m \cos 2\theta \tag{192}$$

$$L_{bb} = L_s + L_m \cos 2\left(\theta - \frac{2\pi}{3}\right) \tag{193}$$

$$L_{cc} = L_s + L_m \cos 2\left(\theta + \frac{2\pi}{3}\right) \tag{194}$$

$$L_{ab} = L_{ba} = -M_s - L_m \cos 2\left(\theta + \frac{\pi}{6}\right) \tag{195}$$

$$L_{bc} = L_{cb} = -M_s - L_m \cos 2\left(\theta - \frac{\pi}{2}\right) \tag{196}$$

$$L_{ca} = L_{ac} = -M_s - L_m \cos 2\left(\theta + \frac{5\pi}{6}\right), \tag{197}$$

$$L_s < L_m, \quad |M_s| > L_m, \tag{198}$$

L_s, L_m and M_s are constants.
Neglecting saturation and slot effect gives

$$L_{FF} = L_F \quad L_{DD} = L_D \quad L_{QQ} = L_Q \tag{199}$$

and the mutual inductances in the rotor are

$$L_{FD} = L_{DF} = M_R, \quad L_{FQ} = L_{QF} = 0, \quad L_{DQ} = L_{QD} = 0. \tag{200}$$

Next are the stator-to-rotor mutual inductances which are namely

$$\begin{aligned}
L_{aF} &= L_{Fa} &&= M_F \cos\theta && (201)\\
L_{bF} &= L_{Fb} &&= M_F \cos\left(\theta - \frac{2\pi}{3}\right) && (202)\\
L_{aF} &= L_{Fa} &&= M_F \cos\left(\theta + \frac{2\pi}{3}\right) && (203)\\
L_{aD} &= L_{Da} &&= M_D \cos\theta && (204)\\
L_{bD} &= L_{Db} &&= M_D \cos\left(\theta - \frac{2\pi}{3}\right) && (205)\\
L_{aD} &= L_{Da} &&= M_D \cos\left(\theta + \frac{2\pi}{3}\right) && (206)\\
L_{aQ} &= L_{Qa} &&= M_Q \cos\theta && (207)\\
L_{bQ} &= L_{Qb} &&= M_Q \cos\left(\theta - \frac{2\pi}{3}\right) && (208)\\
L_{aQ} &= L_{Qa} &&= M_Q \cos\left(\theta + \frac{2\pi}{3}\right) && (209)
\end{aligned}$$

From Equation 191 multiplying both sides by

$$\begin{bmatrix} P & 0 \\ 0 & U_3 \end{bmatrix} \tag{210}$$

and get

$$\begin{bmatrix} P & 0 \\ 0 & U_3 \end{bmatrix} \begin{bmatrix} \lambda_{abc} \\ \lambda_{FDQ} \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & U_3 \end{bmatrix} \begin{bmatrix} L_{aa} & L_{aR} \\ L_{Ra} & L_{RR} \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & U_3 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & U_3 \end{bmatrix} \begin{bmatrix} i_{abc} \\ i_{FDQ} \end{bmatrix} \tag{211}$$

where U_3 is a 3×3 unit matrix and P is the Park's transformation, L_{aa} stator-stator inductances, L_{aR}, L_{Ra} stator-rotor inductances and L_{RR} rotor-rotor inductances. In other words

$$\left[\begin{array}{c} \lambda_0 \\ \lambda_d \\ \lambda_q \\ \hline \lambda_F \\ \lambda_D \\ \lambda_Q \end{array}\right] = \left[\begin{array}{ccc|ccc} L_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_d & 0 & kM_F & kM_D & 0 \\ 0 & 0 & L_q & 0 & 0 & kM_Q \\ \hline 0 & kM_F & 0 & L_F & M_R & 0 \\ 0 & kM_D & 0 & M_R & L_D & 0 \\ 0 & 0 & kM_Q & 0 & 0 & L_Q \end{array}\right] \left[\begin{array}{c} i_0 \\ i_d \\ i_q \\ \hline i_F \\ i_D \\ i_Q \end{array}\right], \tag{212}$$

where

$$L_d = L_s + M_s + \frac{3}{2}L_m \tag{213}$$

$$L_q = L_s + M_s - \frac{3}{2}L_m \tag{214}$$

$$L_0 = L_s - 2M_s \tag{215}$$

$$k = \sqrt{\frac{3}{2}} \tag{216}$$

If we the base for stator quantities are taken as

$$S_{base} = S_R = \text{stator rated VA/phase, VA rms} \tag{217}$$

$$V_{base} = V_R = \text{stator rated line-to-neutral voltage, V rms} \tag{218}$$

$$\omega_{base} = \omega_R = \text{generator rated speed, elec rad/s} \tag{219}$$

then [AF77]

$$\begin{bmatrix} v_0 \\ v_d \\ v_q \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{3}V\sin\alpha \\ \sqrt{3}V\cos\alpha \end{bmatrix} \tag{220}$$

Realize this system with the following values of parameters,

$$\begin{array}{ccccc} L_q = 1.2, & L_d = 1.8, & k = 1.3, & M_F = 0.2, & M_D = 0.4, \\ L_F = 1.2, & M_R = 0.3, & L_D = 1.4, & M_Q = 0.2, & L_Q = 1.1, \\ r = 1.2, & r_F = 1.1, & r_D = 1.4, & r_Q = 1.25, & \omega = 0.34, \\ \tau = 0.025, & T_m = 1, & D = 1.3, & V = 1, & \alpha = \frac{2\pi}{3}, \\ V_F = 1. & & & & \end{array}$$

And the initial conditions were

$$(x_i, x_j) = \mathcal{I}_1 \times \mathcal{I}_1, \quad \mathcal{I}_1 \in \mathcal{I},\ -5 < \mathcal{I}_1 < 5,\ i, j = 1, 2, \ldots, 7,\ i \neq j, x_p = 1,\ p \neq i,\ p \neq j.$$

Hhere I let $x_i, i = 1, 2, \ldots, 7$ represent i_d, i_F, i_D, i_q, i_Q, ω, and δ respectively. The response is shown in Figure 66. ($t_{CPU,average} = 0.0575\,\text{sec}$, averaged for each set of initial states)

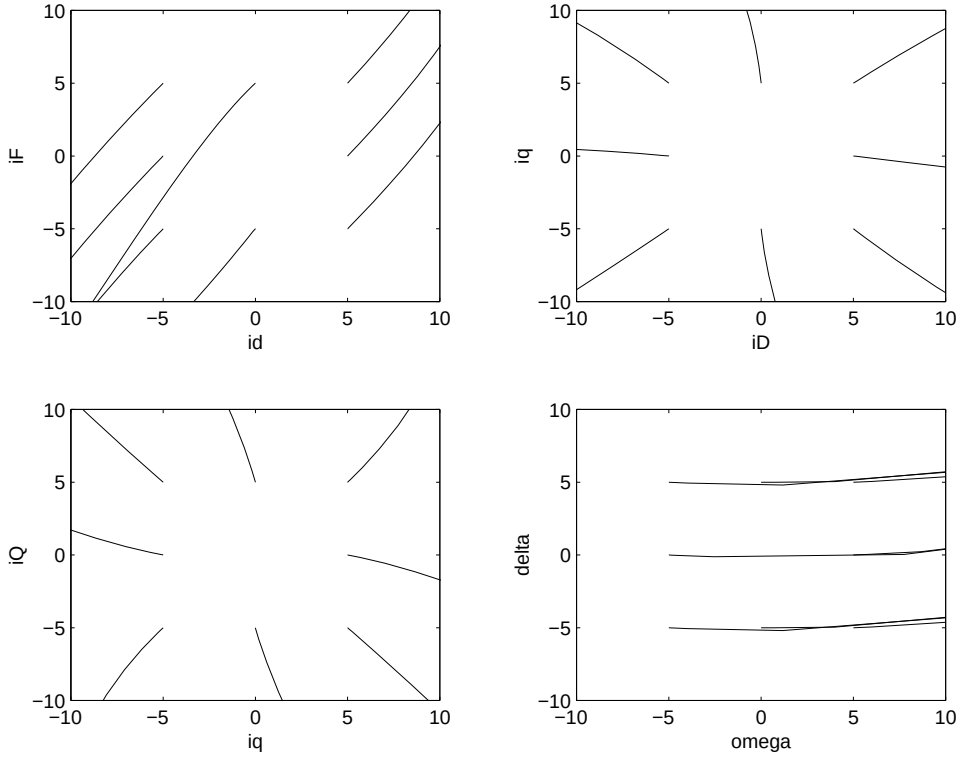


Figure 66: *Synchronous generator* $L_q = 1.2$, $L_d = 1.8$, $k = 1.3$, $M_F = 0.2$, $M_D = 0.4$, $L_F = 1.2$, $M_R = 0.3$, $L_D = 1.4$, $M_Q = 0.2$, $L_Q = 1.1$, $r = 1.2$, $r_F = 1.1$, $r_D = 1.4$, $r_Q = 1.25$, $\omega = 0.34$, $\tau = 0.025$, $T_m = 1$, $D = 1.3$, $V = 1$, $\alpha = \frac{2\pi}{3}$, $V_F = 1$

The flux linkage state-space model when saturation is neglected is

$$\left[\begin{array}{c} \dot{\lambda}_d \\ \dot{\lambda}_F \\ \dot{\lambda}_D \\ \hline \dot{\lambda}_q \\ \dot{\lambda}_Q \\ \hline \dot{\omega} \\ \dot{\delta} \end{array}\right] = [\cdot] \left[\begin{array}{c} \lambda_d \\ \lambda_F \\ \lambda_D \\ \hline \lambda_q \\ \lambda_Q \\ \hline \omega \\ \delta \end{array}\right] + \left[\begin{array}{c} -v_d \\ v_F \\ 0 \\ \hline -v_q \\ 0 \\ \hline \frac{T_m}{\tau_j} \\ \hline -1 \end{array}\right], \tag{221}$$

where $[\cdot]$ is

$$\left[\begin{array}{ccc|cc|cc} -\frac{r}{l_d}\left(1-\frac{L_{MD}}{l_d}\right) & \frac{r}{l_d}\frac{L_{MD}}{l_F} & \frac{r}{l_d}\frac{L_{MD}}{l_D} & -\omega & 0 & 0 & 0 \\ \frac{r_F}{l_F}\frac{L_{MD}}{l_d} & -\frac{r_F}{l_F}\left(1-\frac{L_{MD}}{l_F}\right) & \frac{r_F}{l_F}\frac{L_{MD}}{l_D} & 0 & 0 & 0 & 0 \\ \frac{r_D}{l_D}\frac{L_{MD}}{l_F} & \frac{r_D}{l_D}\frac{L_{MD}}{l_F} & -\frac{r_D}{l_D}\left(1-\frac{L_{MD}}{l_D}\right) & 0 & 0 & 0 & 0 \\ \hline \omega & 0 & 0 & -\frac{r}{l_q}\left(1-\frac{L_{MQ}}{l_q}\right) & \frac{r}{l_q}\frac{L_{MQ}}{l_Q} & 0 & 0 \\ 0 & 0 & 0 & \frac{r_Q}{l_Q}\frac{L_{MQ}}{l_q} & -\frac{r_Q}{l_Q}\left(1-\frac{L_{MQ}}{l_Q}\right) & 0 & 0 \\ \hline -\frac{L_{MD}}{3\tau_j l_d^2}\lambda_q & -\frac{L_{MD}}{3\tau_j l_d l_F}\lambda_Q & -\frac{L_{MD}}{3\tau_j l_d l_D}\lambda_q & \frac{L_{MQ}}{3\tau_j l_q^2}\lambda_d & \frac{L_{MQ}}{3\tau_j l_q l_Q}\lambda_d & -\frac{D}{\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \tag{222}$$

Using the data from [AF77] (Example 4.1 to 4.4) let a 60-Hz synchronous machine have specifications

Rated MVA $= 160\,\text{MVA}$	Rated voltage $= 15\,\text{kV}$, Y connected
Excitation voltage $= 375\,\text{V}$	Stator current $= 6158.40\,\text{A}$
Field current $= 926\,\text{A}$	Power factor $= 0.85$
$L_d = 6.341 \times 10^{-3}\,\text{H}$	$L_F = 2.189\,\text{H}$
$L_D = 5.989 \times 10^{-3}\,\text{H}$	$L_q = 6.118 \times 10^{-3}\,\text{H}$
$L_Q = 1.423 \times 10^{-3}\,\text{H}$	$l_d = l_q(\text{unsaturated}) = 0.5595 \times 10^{-3}\,\text{H}$
$kM_D = 5.782 \times 10^{-3}\,\text{H}$	$kM_Q = 2.779 \times 10^{-3}\,\text{H}$
$r(125\deg C) = 1.542\,\Omega$	$r_F(125\deg C) = 0.371\,\Omega$
$r_D = 18.421 \times 10^{-3}\,\Omega$	$r_Q = 18.969 \times 10^{-3}\,\Omega$
Inertia constant $= 1.765\,\frac{\text{kW.s}}{\text{hp}}$	Field current at rated voltage $= 365\,A$.

The pu quantities are then

$L_d = 1.7$	$L_q = 1.64$	$L_F = 1.65$
$L_D = 1.605$	$L_Q = 1.526$	$kM_F = M_R = kM_D = 1.55$
$l_d = l_q = 0.15$	$kM_Q = 1.49$	$r = 0.001096$
$r_F = 0.000742$	$r_D = 0.0131$	$r_Q 0.0540$
$H = 2.37\,s$		

Let the mechanical damping constant be a typical value of

$$D = 2.$$

The electrical frequency (base value) is

$$\omega_B \quad = 60 \frac{\text{rounds}}{second} \cdot 2\pi \frac{\text{radians}}{\text{round}} \quad \approx 377 \frac{\text{radians}}{second},$$

and provided that the machine runs at approximately the rated speed, the electrical frequency (per-unit value) is

$$\frac{\omega}{\omega_B} \approx 1.$$

The normalized swing equation is

$$\tau_j \frac{d}{dt_u} \omega_u \quad = T_{mu} - T_{eu} \quad = T_{au} \quad \text{pu}. \tag{223}$$

But the normalized per-unit swing equation together with the per-unit torque equation give

$$\frac{2H}{\omega_B} \dot{\omega} = T_a = T_m - T_e \quad \text{pu}(3\phi), \tag{224}$$

where H is the inertial constant. Substituting the per-unit values of time and angular speed

$$t_u = \omega_B t, \quad \omega_u = \frac{\omega}{\omega_B}, \tag{225}$$

and get

$$2H\omega_B \frac{d\omega}{dt_u} = T_{au}. \tag{226}$$

Thus

$$\tau_j = 2H\omega_B \tag{227}$$

when time is in pu.
The result from simulation based on the above parameters are shown in Figure 67. Figure 67 has got initial conditions as follows

top-left $i_d(0) = \{-3 \leq \mathcal{I} \leq 3\}$, $i_F(0) = \pm 3$, $i_D(0) = i_q(0) = i_Q(0) = \omega(0) = \delta(0) = 0$. The state variables in time domain are shown in Figure 68.

top-right $i_D(0) = \{-3 \leq \mathcal{I} \leq 3\}$, $i_q(0) = \pm 3$, $i_d(0) = i_F(0) = i_Q(0) = \omega(0) = \delta(0) = 0$ The state variables in time domain are shown in Figure 69.

bottom-left $i_q(0) = \{-3 \leq \mathcal{I} \leq 3\}$, $i_Q(0) = \pm 3$, $i_d(0) = i_F(0) = i_D(0) = \omega(0) = \delta(0) = 0$ The state variables in time domain are shown in Figure 70.

bottom-right $\omega(0) = \{-3 \leq \mathcal{I} \leq 3\}$, $\delta(0) = \pm 3$, $i_d(0) = i_F(0) = i_D(0) = i_q(0) = i_Q(0) = 0$ The state variables in time domain are shown in Figure 71.

The parameters were

$$\begin{array}{ccccc} L_q = 1.64, & L_d = 1.7, & kM_F = kM_D = M_R = 1.55, & L_F = 1.65, & L_D = 1.605, \\ kM_Q = 1.49, & L_Q = 1.526, & r = 0.001096, & r_F = 0.000742, & r_D = 0.0131, \\ r_Q = 0.0540, & \omega = 1, & T_m = 1, & D = 2, & V = 1, \\ \alpha = \frac{2\pi}{3}, & V_F = 1. & & & \end{array}$$

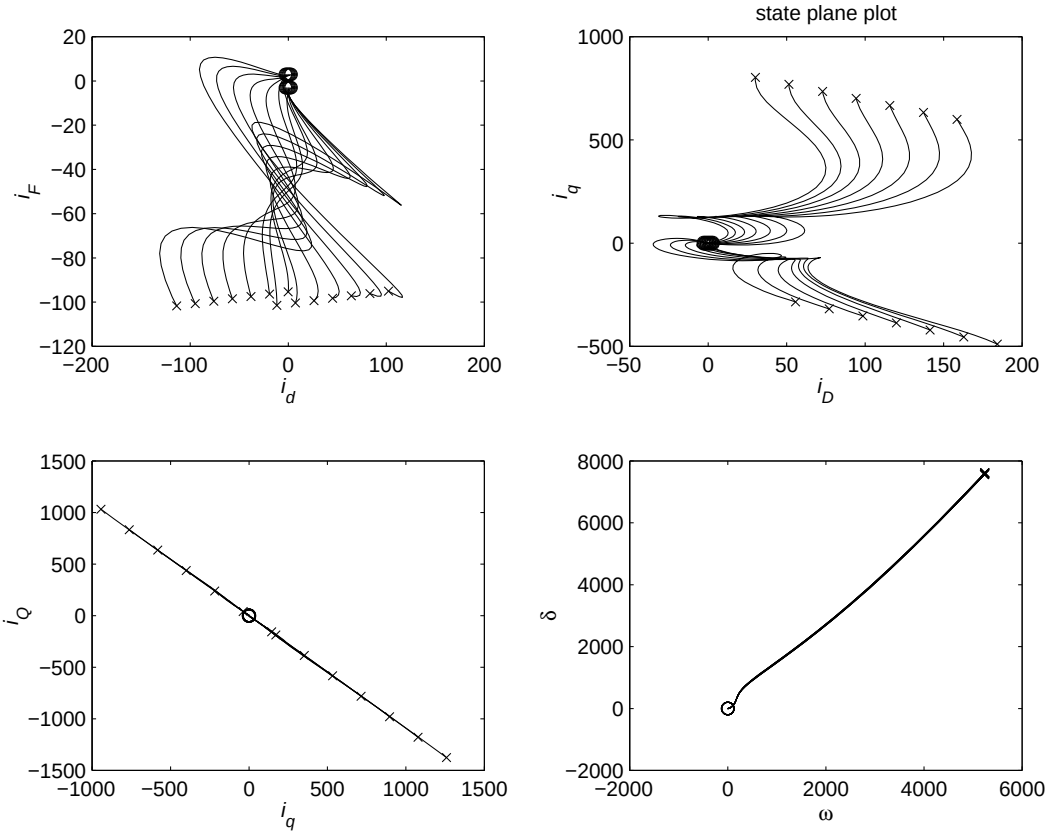


Figure 67: *State plane, synchronous generator (no control)* ($L_q = 1.64$, $L_d = 1.7$, $kM_F = kM_D = M_R = 1.55$, $L_F = 1.65$, $L_D = 1.605$, $kM_Q = 1.49$, $L_Q = 1.526$, $r = 0.001096$, $r_F = 0.000742$, $r_D = 0.0131$, $r_Q = 0.0540$, $\omega = 1$, $T_m = 1$, $D = 2$, $V = 1$, $\alpha = \frac{2\pi}{3}$, $V_F = 1$)

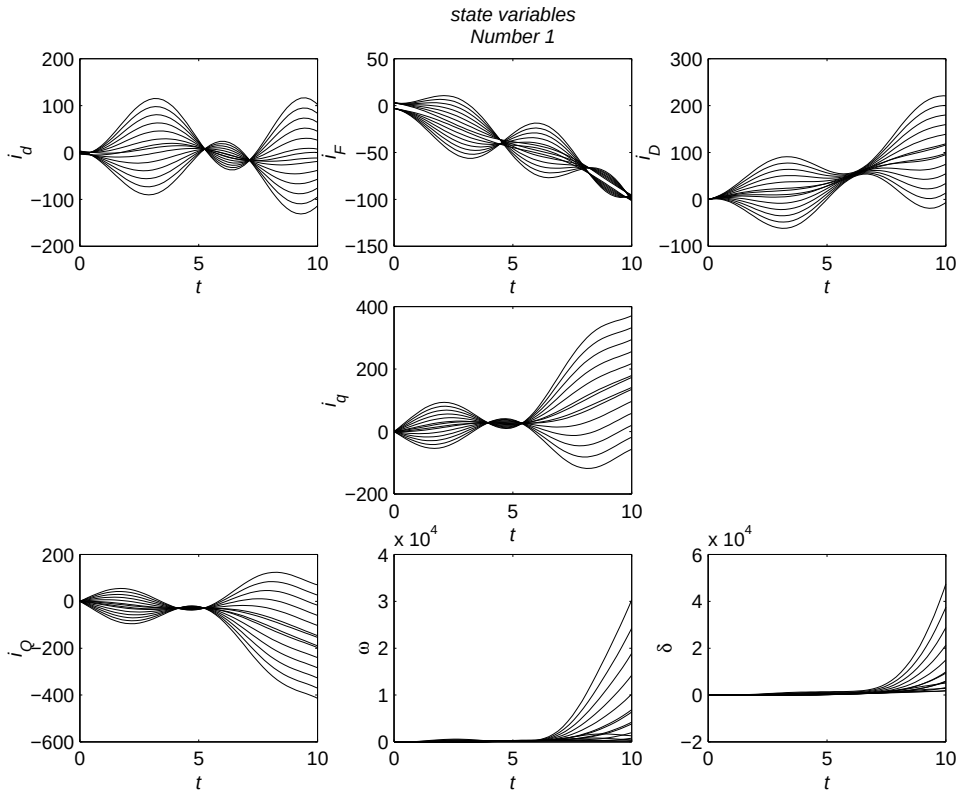


Figure 68: *State variables, synchronous generator (no control) ($L_q = 1.64$, $L_d = 1.7$, $kM_F = kM_D = M_R = 1.55$, $L_F = 1.65$, $L_D = 1.605$, $kM_Q = 1.49$, $L_Q = 1.526$, $r = 0.001096$, $r_F = 0.000742$, $r_D = 0.0131$, $r_Q = 0.0540$, $\omega = 1$, $T_m = 1$, $D = 2$, $V = 1$, $\alpha = \frac{2\pi}{3}$, $V_F = 1$)*

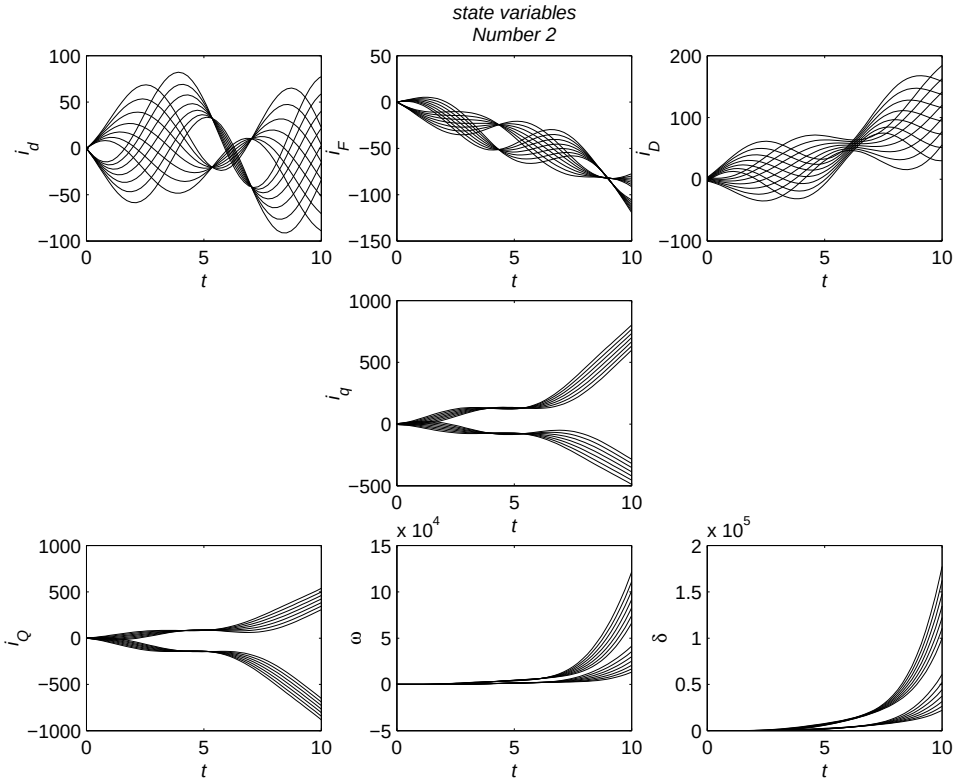


Figure 69: *State variables, synchronous generator (no control) ($L_q = 1.64$, $L_d = 1.7$, $kM_F = kM_D = M_R = 1.55$, $L_F = 1.65$, $L_D = 1.605$, $kM_Q = 1.49$, $L_Q = 1.526$, $r = 0.001096$, $r_F = 0.000742$, $r_D = 0.0131$, $r_Q = 0.0540$, $\omega = 1$, $T_m = 1$, $D = 2$, $V = 1$, $\alpha = \frac{2\pi}{3}$, $V_F = 1$)*

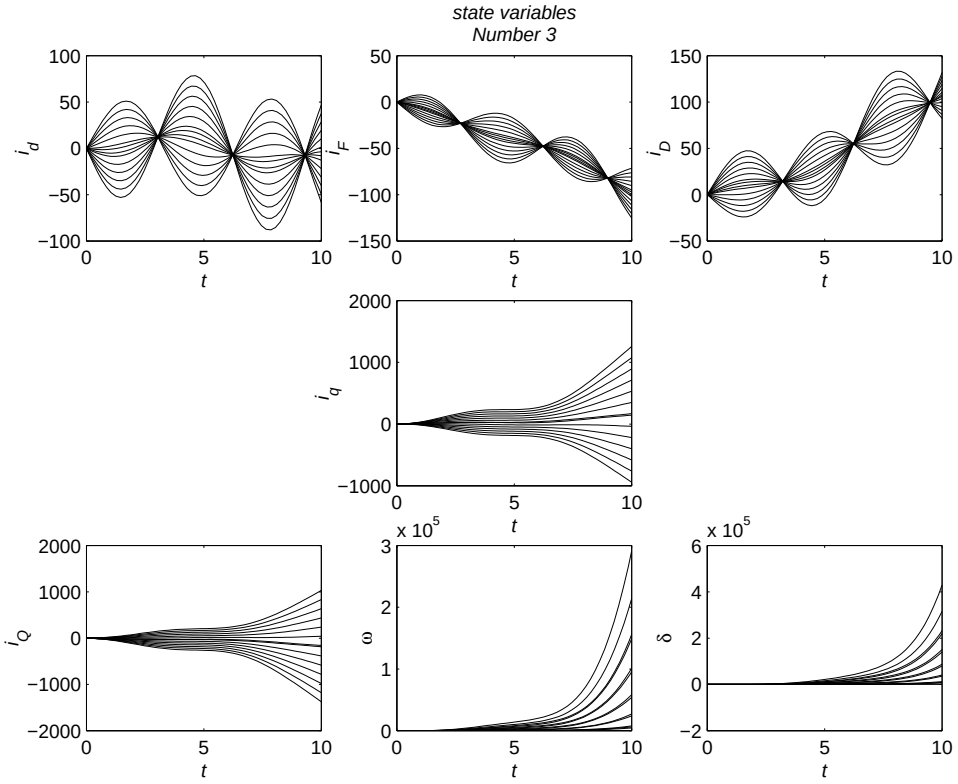


Figure 70: *State variables, synchronous generator (no control) ($L_q = 1.64$, $L_d = 1.7$, $kM_F = kM_D = M_R = 1.55$, $L_F = 1.65$, $L_D = 1.605$, $kM_Q = 1.49$, $L_Q = 1.526$, $r = 0.001096$, $r_F = 0.000742$, $r_D = 0.0131$, $r_Q = 0.0540$, $\omega = 1$, $T_m = 1$, $D = 2$, $V = 1$, $\alpha = \frac{2\pi}{3}$, $V_F = 1$)*

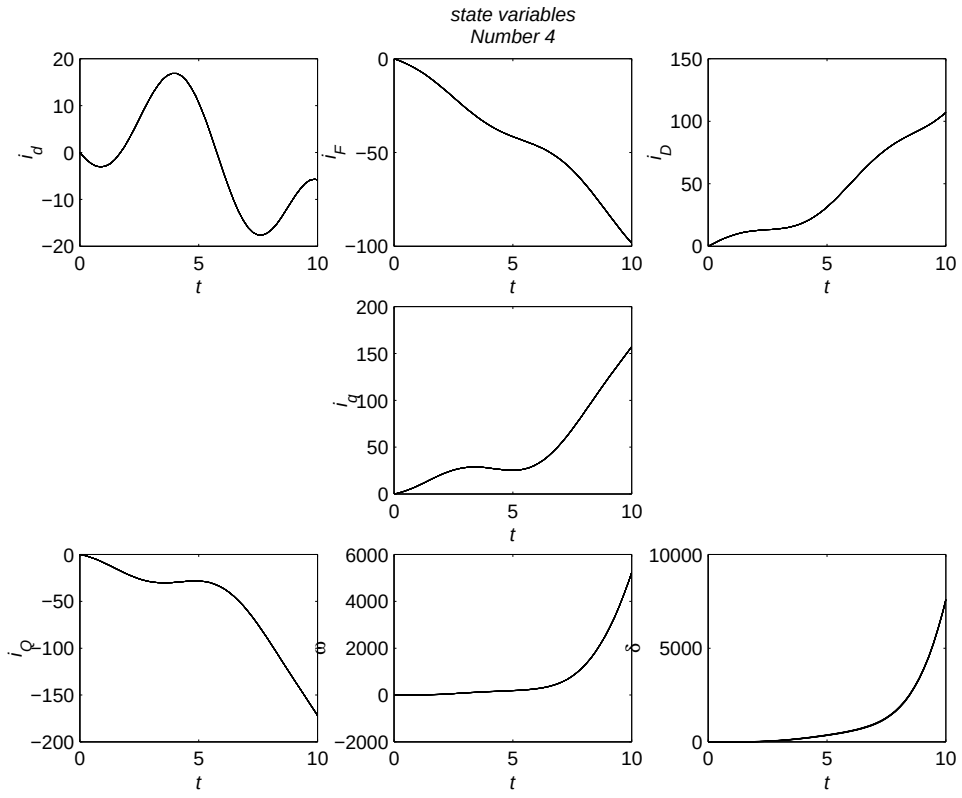


Figure 71: *State variables, synchronous generator (no control)* ($L_q = 1.64$, $L_d = 1.7$, $kM_F = kM_D = M_R = 1.55$, $L_F = 1.65$, $L_D = 1.605$, $kM_Q = 1.49$, $L_Q = 1.526$, $r = 0.001096$, $r_F = 0.000742$, $r_D = 0.0131$, $r_Q = 0.0540$, $\omega = 1$, $T_m = 1$, $D = 2$, $V = 1$, $\alpha = \frac{2\pi}{3}$, $V_F = 1$)

α (alpha)	β (beta)	γ, Γ (gamma)	δ, Δ (delta)	ϵ (epsilon)	ε (varepsilon)
θ, Θ (theta)	ϑ (vartheta)	ι (iota)	κ (kappa)	λ, Λ (lambda)	μ (mu)
π, Π (pi)	ϖ (varpi)	ρ (rho)	ϱ (varrho)	σ, Σ (sigma)	ς (varsigma)
ϕ, Φ (phi)	φ (varphi)	χ (chi)	ψ, Ψ (psi)	ω, Ω (omega)	η (eta)
ζ (zeta)	ξ, Ξ (xi)	ν (nu)	υ, Υ (upsilon)	τ (tau)	

Table 5: *Greek alphabets*

How the Greek symbols are read

In order to allow wider a choice of variables as well as constant names Greek letters are used in addition to Roman alphabets. Table 5 shows how the Greek alphabets are called [Lam86].

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